METR4202 -- Robotics
Tutorial 4 and 5 – Weeks 4 & 5:
Trajectory Generation & Motion Planning

Reading

Please read/review chapter 9 of Robotics, Vision and Control.

Questions

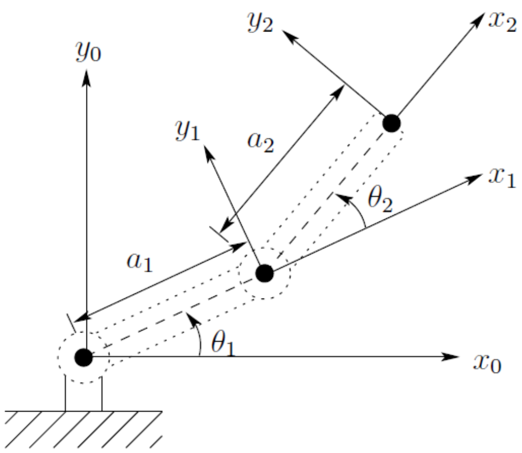

Figure 1: Two DOF Robot manipulator

1. Write the full equation of motion for the 2R arm above (i.e., τ_1 and τ_2 as a function of θ_1 and θ_2 and its derivatives)

Start with the masses of links: m_1 and m_2, to get the Mass Matrix recall (lecture 5)

$$M = \sum_{i=1}^{N}\left(m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T I_{C_i} J_{\omega_i}\right)$$

$$M = m_1 J_{v_1}^T J_{v_1} + J_{\omega_1}^T I_1 J_{\omega_1} + m_2 J_{v_2}^T J_{v_2} + J_{\omega_2}^T I_2 J_{\omega_2}$$

Note that:

m_i = the mass of the i^{th} link

m_{ij} = the ij element of the mass matrix

$$m_{ijk} \equiv \frac{\partial m_{ij}}{\partial q_k}$$

Note this is with respect to the configuration variable, not time.

On that subject, the derivative with respect to time would be: $\frac{d}{dt} m_{ij} = \sum_{k=1}^{N} m_{ijk}\dot{q}_k$

The center of mass of each link is at the joint center, this $l_1 \equiv a_1/2$ and $l_2 \equiv a_2/2$

To compute the Jacobians (J_v and J_ω), we need to calculate the forward kinematics.

Recall that the position vectors (Lec 3, Slide 34) for a 2R arm are:

$${}^{0}P_1 = \begin{bmatrix} a_1C_1 \\ a_1S_1 \\ 0 \end{bmatrix} \quad \text{(this reads as "Position of Frame 1 as seen in 0")}, \quad {}^{0}P_2 = \begin{bmatrix} a_1C_1 + a_2C_{12} \\ a_1S_2 + a_2S_{12} \\ 0 \end{bmatrix}$$

Thus with respect to Frame {0}, the translational velocity Jacobians (i.e., the matrices that encode the differential relationship between joint velocities and workspace tip velocities) are found by direct differentiation of the position vectors ${}^{0}\mathbf{P}_1$ and ${}^{0}\mathbf{P}_2$.

$${}^{0}J_{v_1} = \begin{bmatrix} -a_1S_1 & 0 \\ a_1C_1 & 0 \\ 0 & 0 \end{bmatrix}, {}^{0}J_{v_2} = \begin{bmatrix} -a_1S_1 - a_1S_{12} & -a_2S_{12} \\ a_1C_1 + a_2C_{12} & a_2C_{12} \\ 0 & 0 \end{bmatrix}$$

$$\to m_1 J_{v_1}^T J_{v_1} = \begin{bmatrix} m_1a_1^2 & 0 \\ 0 & 0 \end{bmatrix}, \; m_2 J_{v_2}^T J_{v_2} = \begin{bmatrix} m_2\left(a_1^2 + a_2^2 + 2a_1a_2C_2\right) & m_2\left(a_2^2 + a_1a_2C_2\right) \\ m_2\left(a_2^2 + a_1a_2C_2\right) & m_2a_2^2 \end{bmatrix}$$

The rotational velocity Jacobian matrices with respect to Frame {0} are given by

$$J_{\omega_1} = \left[\bar{\varepsilon}_1\mathbf{z}_1 \quad \mathbf{0}\right], J_{\omega_2} = \left[\bar{\varepsilon}_1\mathbf{z}_1 \quad \bar{\varepsilon}_2\mathbf{z}_2\right]$$

As both joints are revolute (ε=0), these matrices are $J_{\omega_1} = \left[\mathbf{z}_1 \quad \mathbf{0}\right], J_{\omega_2} = \left[\mathbf{z}_1 \quad \mathbf{z}_2\right]$

Thus, $J_{\omega_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, J_{\omega_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$, and after some substitution and simplification we have

$J_{\omega_1}^T I_1 J_{\omega_1} = \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix}, J_{\omega_2}^T I_2 J_{\omega_2} = \begin{bmatrix} I_2 & I_2 \\ I_2 & I_2 \end{bmatrix}$ where I is about the *z*-axis ($I_1 = I_{\{zz\}1}$ and $I_2 = I_{\{zz\}2}$)

Finally, the mass matrix, M is

$$M = \begin{bmatrix} m_1a_1^2 + I_1 + m_2\left(a_1^2 + a_2^2 + 2a_1a_2C_2\right) + I_2 & m_2\left(a_2^2 + a_1a_2C_2\right) + I_2 \\ m_2\left(a_2^2 + a_1a_2C_2\right) + I_2 & m_2a_2^2 + I_2 \end{bmatrix}$$

The Centrifugal and Coriolis Matrix **v** is found directly by recalling Christoffel symbols (please review Christoffel symbols from dynamics and the mass notation from the previous page)

$b_{i,jk} = \frac{1}{2}\left(m_{ijk} + m_{ikj} - m_{jki}\right)$ and with $b_{iii} = b_{iji} = 0$,

the Centrifugal matrix becomes

$$B = \begin{bmatrix} 2b_{112} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}m_{112} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(\frac{\partial m_{11}}{\partial \theta_2}\right) \\ 0 \end{bmatrix} = \begin{bmatrix} -m_2a_1a_2S_2 \\ 0 \end{bmatrix},$$

and the Coriolis matrix can be written as

$$C = \begin{bmatrix} 0 & b_{122} \\ b_{211} & 0 \end{bmatrix} = \begin{bmatrix} 0 & m_{122} \\ -\frac{1}{2}m_{112} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \left(\frac{\partial m_{12}}{\partial \theta_2}\right) \\ -\frac{1}{2}\left(\frac{\partial m_{11}}{\partial \theta_2}\right) & 0 \end{bmatrix} = \begin{bmatrix} 0 & -m_2a_1a_2S_2 \\ m_2a_1a_2S_2 & 0 \end{bmatrix}$$

Summing this together gives

$$V = \begin{bmatrix} -m_2a_1a_2S_2 \\ 0 \end{bmatrix}\left(\dot{\theta}_1\dot{\theta}_2\right) + \begin{bmatrix} 0 & -m_2a_1a_2S_2 \\ m_2a_1a_2S_2 & 0 \end{bmatrix}\begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix}$$

The next factor to consider is gravity.
While the problem does not specify a gravity direction, we assume it is acting parallel to the *y*-axis. This gives $\mathbf{g} = \begin{bmatrix} 0 & -g & 0 \end{bmatrix}$. (Note that if we latter wish to assume that gravity is acting along the *z*-axis (into the page), this could be treated by setting $\mathbf{g} = \begin{bmatrix} 0 & 0 & -g \end{bmatrix}$)
With respect to Frame {0}, the gravity vector can be calculated as

$$\mathbf{G} = -\left[J_{v_{C1}}^T m_{C1}\mathbf{g} + J_{v_{C2}}^T m_{C2}\mathbf{g} \right]$$

However, we have to be careful because the gravity acts at the mass center (which is represented by the notation C1 and C2). Again, recall that we have $l_1=a_1/2$ and $l_2=a_2/2$
Given the structure of the problem, the Jacbobians are be determined by inspection.
Thus,

$${}^0G = -\begin{bmatrix} -l_1S_1 & l_1C_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -m_1 g \\ 0 \end{bmatrix} - \begin{bmatrix} -a_1S_1 - l_2S_{12} & a_1C_1 + l_2C_{12} & 0 \\ -l_2S_{12} & l_2C_{12} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -m_2 g \\ 0 \end{bmatrix}$$

$${}^0G = \begin{bmatrix} (m_1l_1 + m_2a_1)C_1 + m_2l_2C_{12} \\ m_2l_2C_{12} \end{bmatrix}(g) = \begin{bmatrix} \left(\frac{1}{2}m_1 + m_2\right)a_1C_1 + \frac{1}{2}m_2a_2C_{12} \\ \frac{1}{2}m_2a_2C_{12} \end{bmatrix}(g)$$

The **Equations of Motion** can be found by putting these terms together to give
(for review see also Lecture 4, Slide 30 and Lecture 5, Slide 7)

$$\boldsymbol{\tau} = M(\theta)\ddot{\boldsymbol{\theta}} + \mathbf{v}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \mathbf{g}(\boldsymbol{\theta})$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} m_1a_1^2 + I_1 + m_2\left(a_1^2 + a_2^2 + 2a_1a_2C_2\right) + I_2 & m_2\left(a_2^2 + a_1a_2C_2\right) + I_2 \\ m_2\left(a_2^2 + a_1a_2C_2\right) + I_2 & m_2a_2^2 + I_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$+\left(\dot{\theta}_1\dot{\theta}_2\right)\begin{bmatrix} -m_2a_1a_2S_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -m_2a_1a_2S_2 \\ m_2a_1a_2S_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix} + (g)\begin{bmatrix} \left(\frac{1}{2}m_1 + m_2\right)a_1C_1 + \frac{1}{2}m_2a_2C_{12} \\ \frac{1}{2}m_2a_2C_{12} \end{bmatrix}$$