Forward Kinematics

METR 4202: **Robotics** & Automation

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August 2, 2017

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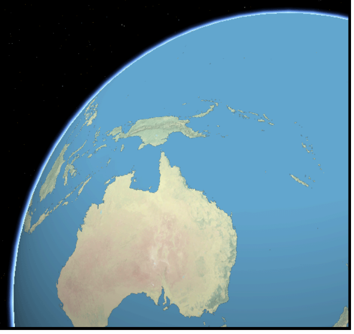
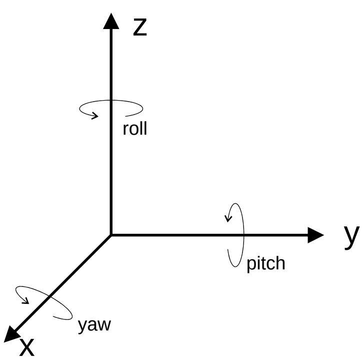
Lecture Schedule

Week	Date	Lecture (W: 3:05p-4:50, 7-222)
1	26-Jul	Introduction + Representing Position & Orientation & State
2	2-Aug	Robot Forward Kinematics (Frames, Transformation Matrices & Affine Transformations)
3	9-Aug	Robot Inverse Kinematics & Dynamics (Jacobeans)
4	16-Aug	*Ekka Day* (Robot Kinematics & Kinetics Review)
5	23-Aug	Robot Sensing: Perception & Linear Observers
6	30-Aug	Robot Sensing: Single View Geometry & Lines
7	6-Sep	Robot Sensing: Multiple View Geometry
8	13-Sep	Robot Sensing: Feature Detection
9	20-Sep	***Mid-Semester Exam***
	27-Sep	*Study break*
10	4-Oct	Motion Planning
11	11-Oct	Probabilistic Robotics: Localization & SLAM
12	18-Oct	Probabilistic Robotics: Planning & Control (State-Space/Shaping the Dynamic Response/LQR)
13	25-Oct	The Future of Robotics/Automation + Challenges + Course Review

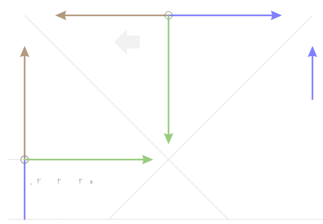
First thing about structure
→ **Space**



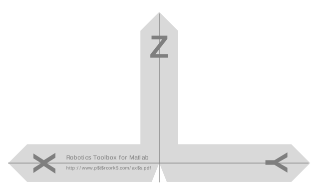
Today's Lecture is about:
Frames & Their Mathematics



- Make one (online):
  - SpnS Template

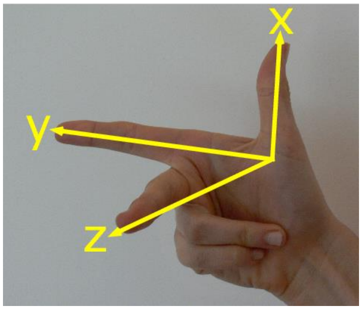


  - Peter Corke's template



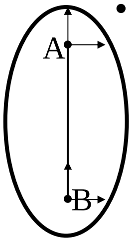
Don’t Confuse a Frame with a Point

- Points
  - Position Only – Doesn’t Encode Orientation
- Frame
  - Encodes both position and orientation
  - Has a “handedness”

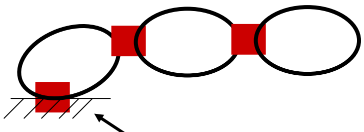


Kinematics Definition

- **Kinematics**: The study of motion in space (without regard to the forces which cause it)



- Assume:
  - Points with ***right-hand Frames***
  - *Rigid-bodies* in 3D-space (6-dof)
  - 1-dof joints: Rotary (R) or Prismatic (P) (5 constraints)

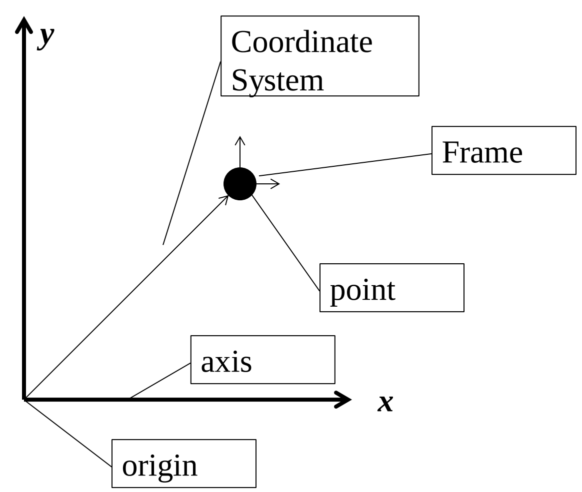


N links
M joints
→DOF = 6N-5M
➔ If N=M, then DOF=N.

Kinematics

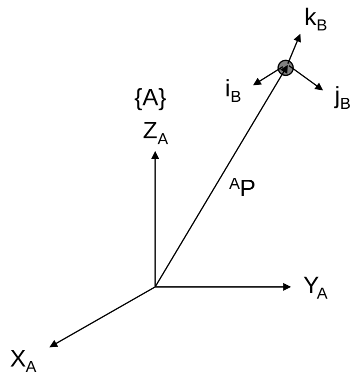
- Kinematic modelling is one of the most important analytical tools of robotics.
- Used for modelling mechanisms, actuators and sensors
- Used for on-line control and off-line programming and simulation
- In mobile robots kinematic models are used for:
  - steering (control, simulation)
  - perception (image formation)
  - sensor head and communication antenna pointing
  - world modelling (maps, object models)
  - terrain following (control feedforward)
  - gait control of legged vehicles

Basic Terminology



Coordinate System

- The position and orientation as specified only make sense with respect to some coordinate system



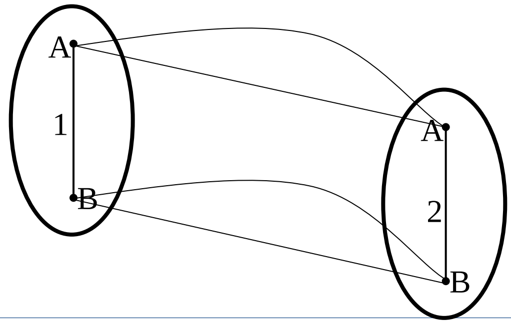
Frames of Reference

- A frame of reference defines a coordinate system relative to some point in space
- It can be specified by a position and orientation relative to other frames
- The *inertial frame* is taken to be a point that is assumed to be fixed in space

- Two types of motion:
  - Translation
  - Rotation

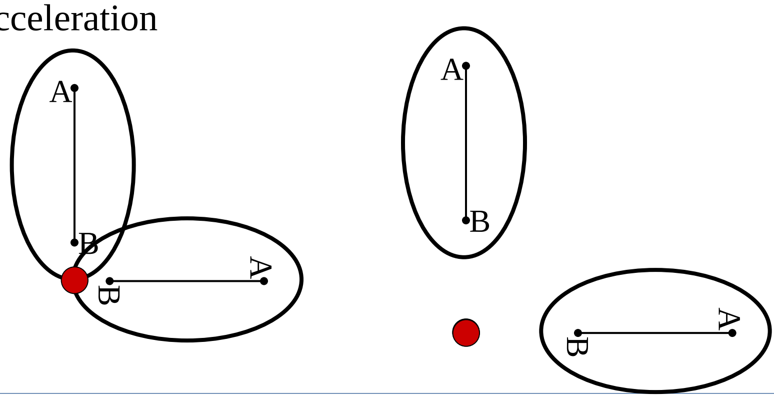
Translation

- A motion in which a straight line with in the body keeps the same direction during the
  - **Rectilinear Translation:** Along straight lines
  - **Curvilinear Translation**: Along curved lines



Rotation

- The particles forming the rigid body move in parallel planes along circles centered around the same fixed axis (called the **axis of rotation**).
- Points on the axis of rotation have zero velocity and acceleration



Rotation: Representations

- Orientation are not "Cartesian"
  - Non-commutative
  - Multiple representations

- Some representations:
  - **Rotation Matrices**: Homegenous Coordinates
  - Euler Angles: 3-sets of rotations in sequence
  - Quaternions: a 4-paramameter representation that exploits ½ angle properties
  - Screw-vectors (from Charles Theorem) : a canonical representation, its reciprocal is a "wrench" (forces)

Euler Angles

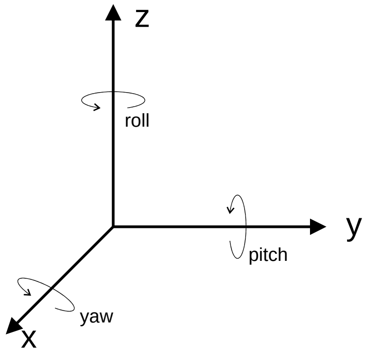
- Minimal representation of orientation (α,β,γ)
- Represent a rotation about an axis of a **moving** coordinate frame
  → ${}^{A}_{B}\mathbf{R}$: Moving frame **B** w/r/t fixed A
- The location of the axis of each successive rotation depends on the previous one! …
- So, Order Matters (12 combinations, why?)
- Often Z-Y-X:
  - α: rotation about the **z** axis
  - β: rotation about the rotated **y** axis
  - γ: rotation about the twice rotated **x** axis
- Has singularities! … (e.g., β=±90°)

Fixed Angles

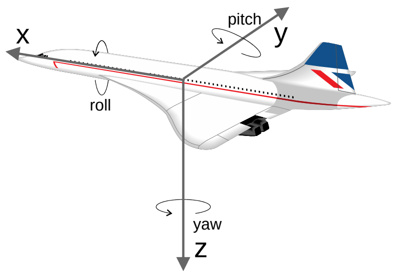
- Represent a rotation about an axis of a **fixed** coordinate frame.
- Again 12 different orders
- Interestingly:
  3 rotations about 3 axes of a **fixed** frame define the same orientation as the same 3 rotations taken in the **opposite order** of the **moving** frame
- For X-Y-Z:
  - ψ: rotation about $\mathbf{x}_A$ (sometimes called "yaw")
  - θ: rotation about $\mathbf{y}_A$ (sometimes called "pitch")
  - φ: rotation about $\mathbf{z}_A$ (sometimes called "roll")

Roll – Pitch – Yaw

- In many Kinematics References:



- In many Engineering Applications:



→ Be careful:
This name is given to other conventions too!

Euler Angles [1]: **X-Y-Z Fixed Angles** (Roll-Pitch-Yaw)

- One method of describing the orientation of a Frame {B} is:
  - Start with the frame coincident with a known reference {A}. Rotate {B} first about X_A by an angle γ, then about Y_A by an angle β and finally about Z_A by an angle α.

$$
\begin{aligned}
{}^{A}R_{BXYZ}(\gamma,\beta,\alpha) &= R_Z(\alpha)R_Y(\beta)R_X(\gamma) \\
&= \begin{bmatrix} c_\alpha & -s_\alpha & 0 \\ s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\beta & 0 & s_\beta \\ 0 & 1 & 0 \\ -s_\beta & 0 & c_\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\gamma & -s_\gamma \\ 0 & s_\gamma & c_\gamma \end{bmatrix} \\
&= \begin{bmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}
\end{aligned}
$$

Euler Angles [2]: Z-Y-X Euler Angles

- Another method of describing the orientation of {B} is:
  - Start with the frame coincident with a known reference {A}. Rotate {B} first about Z_B by an angle α, then about Y_B by an angle β and finally about X_B by an angle γ.

$$
\begin{aligned}
{}^{A}R_{BZ'Y'X'}(\gamma,\beta,\alpha) &= R_Z(\alpha)R_Y(\beta)R_X(\gamma) \\
&= \begin{bmatrix} c_\alpha & -s_\alpha & 0 \\ s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\beta & 0 & s_\beta \\ 0 & 1 & 0 \\ -s_\beta & 0 & c_\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\gamma & -s_\gamma \\ 0 & s_\gamma & c_\gamma \end{bmatrix} \\
&= \begin{bmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}
\end{aligned}
$$

Unit Quaternion (ϵ_0, ϵ_1, ϵ_2, ϵ_3) [1]

- Does not suffer from singularities

$$\epsilon \equiv \epsilon_0 + \left(\epsilon_1 \hat{\mathbf{i}} + \epsilon_2 \hat{\mathbf{j}} + \epsilon_3 \hat{\mathbf{k}}\right)$$

- Uses a "4-number" to represent orientation

$$ii = jj = kk = -1$$
$$ij = k, jk = i, ki = j, ji = -k, kj = -1, ik = -j$$

- Product:

$$\begin{aligned} \mathbf{ab} &= (a_0 b_0 - a_1 b_1 - a_2 b_2 + a_3 b_3) \\ &+ (a_0 b_1 + a_1 b_0 + a_2 b_3 - a_3 b_2)\,\hat{i} \\ &+ (a_0 b_2 + a_2 b_0 + a_3 b_1 + a_1 b_3)\,\hat{j} \\ &+ (a_0 b_3 + a_3 b_0 + a_1 b_2 - a_2 b_1)\,\hat{k} \end{aligned}$$

- Conjugate:

$$\tilde{\epsilon} \equiv \epsilon_0 - \epsilon_1 \hat{\mathbf{i}} - \epsilon_2 \hat{\mathbf{j}} - \epsilon_3 \hat{\mathbf{k}}$$
$$\epsilon\tilde{\epsilon} = \tilde{\epsilon}\epsilon = \epsilon_0^2 + \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2$$

Unit Quaternion [2]: Describing Orientation

- Set $\epsilon_0 = 0$
  Then $\mathbf{p}=(p_x,p_y,p_z)$ → $\mathbf{p} = p_x \hat{\mathbf{i}} + p_y \hat{\mathbf{j}} + p_z \hat{\mathbf{k}}$
- Then given ϵ
  the operation $\epsilon \mathbf{p} \tilde{\epsilon}$: rotates **p** about (ϵ_1, ϵ_2, ϵ_3)
- Unit Quaternion → Rotation Matrix

$$\mathbf{R} = \begin{pmatrix} 1 - 2\left(\epsilon_2^2 + \epsilon_3^2\right) & 2\left(\epsilon_1\epsilon_2 - \epsilon_0\epsilon_3\right) & 2\left(\epsilon_1\epsilon_3 - \epsilon_0\epsilon_2\right) \\ 2\left(\epsilon_1\epsilon_2 - \epsilon_0\epsilon_3\right) & 1 - 2\left(\epsilon_1^2 + \epsilon_3^2\right) & 2\left(\epsilon_2\epsilon_3 - \epsilon_0\epsilon_1\right) \\ 2\left(\epsilon_1\epsilon_3 - \epsilon_0\epsilon_2\right) & 2\left(\epsilon_2\epsilon_3 - \epsilon_0\epsilon_1\right) & 1 - 2\left(\epsilon_1^2 + \epsilon_2^2\right) \end{pmatrix}$$

Direction Cosine

- Uses the Direction Cosines (read dot products) of the Coordinate Axes of the moving frame with respect to the fixed frame

$$^{A}\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$$

$$^{A}\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$$

$$^{A}\mathbf{w} = w_x\mathbf{i} + w_y\mathbf{j} + w_z\mathbf{k}$$

- It forms a rotation matrix!

$$\begin{array}{c|ccc} {}^{A}_{B}R & (b_x)\,\hat{\imath}_B & (b_y)\,\hat{\jmath}_B & (b_z)\,\hat{k}_B \\ \hline (a_x)\,\hat{\imath}_A & \hat{\imath}_B\cdot\hat{\imath}_A & \hat{\jmath}_B\cdot\hat{\imath}_A & \hat{k}_B\cdot\hat{\imath}_A \\ (a_y)\,\hat{\jmath}_A & \hat{\imath}_B\cdot\hat{\jmath}_A & \hat{\jmath}_B\cdot\hat{\jmath}_A & \hat{k}_B\cdot\hat{\jmath}_A \\ (a_z)\,\hat{k}_A & \hat{\imath}_B\cdot\hat{k}_A & \hat{\jmath}_B\cdot\hat{k}_A & \hat{k}_B\cdot\hat{k}_A \end{array}$$

Screw Displacements

- Comes from the notion that all motion can be viewed as a rotation (Rodrigues formula)
- Define a vector along the axis of motion (screw vector)
  - Rotation (screw angle)
  - Translation (pitch)
  - Summations → via the screw triangle!

Generalizing

Special Orthogonal & Special Euclidean Lie Algebras

- SO(n): R**o**tations

$$SO(n) = \{R \in \mathbb{R}^{n\times n} : RR^T = I, \det R = +1\}.$$

$$\exp(\widehat{\omega}\theta) = e^{\widehat{\omega}\theta} = I + \theta\widehat{\omega} + \frac{\theta^2}{2!}\widehat{\omega}^2 + \frac{\theta^3}{3!}\widehat{\omega}^3 + \ldots$$

- SE(n): Transformations of EUCLIDEAN space

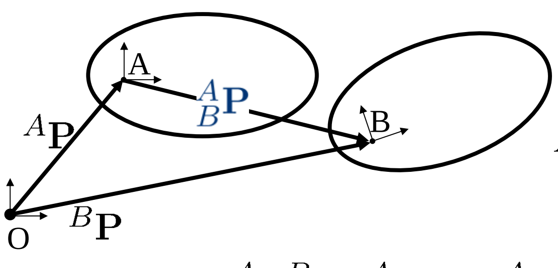
$$SE(n) := \mathbb{R}^n \times SO(n).$$

$$SE(3) = \{(p, R) : p \in \mathbb{R}^3, R \in SO(3)\} = \mathbb{R}^3 \times SO(3).$$

Position and Orientation [1]

- A **position** vectors specifies the location of a **point** in 3D (Cartesian) space

$$\mathbf{P} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$



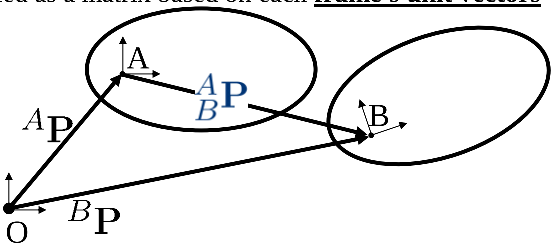
$${}^{A}\mathbf{P} + {}^{A}\mathbf{P}^{B} - {}^{B}\mathbf{P} = 0$$

$${}^{A}\mathbf{P}^{B} = {}^{A}\mathbf{P}_{B} = {}^{A}_{B}\mathbf{P} = \begin{bmatrix} {}^{B}p_x \\ {}^{B}p_y \\ {}^{B}p_z \end{bmatrix} - \begin{bmatrix} {}^{A}p_x \\ {}^{A}p_y \\ {}^{A}p_z \end{bmatrix}$$

- BUT we **also** concerned with its orientation in 3D space. This is specified as a matrix based on each **frame's unit vectors**

Position and Orientation [2]

- Orientation in 3D space:
  This is specified as a matrix based on each **frame's unit vectors**



- Describes {B} relative to {A}
  → The orientation of frame {B} relative to coordinate frame {A}
- Written "from {A} to {B}" or "given {A} getting to {B}"

$$^{A}\mathbf{R}_{B} = {}^{A}_{B}\mathbf{R} = \left[\begin{array}{ccc} ^{A}\hat{\imath}_{B} & ^{A}\hat{\jmath}_{B} & ^{A}\hat{k}_{B} \end{array}\right]$$

- **Columns** are **{B} written in {A}**

Position and Orientation [3]

- The rotations can be analysed based on the unit components …
- That is: the components of the orientation matrix are the unit vectors projected **onto** the unit directions of the reference frame

$$^{A}_{B}\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\begin{array}{c|c} {}^{A}_{B}R & \begin{array}{ccc} (b_x)\,\hat{\imath}_B & (b_y)\,\hat{\jmath}_B & (b_z)\,\hat{k}_B \end{array} \\ \hline \begin{array}{c} (a_x)\,\hat{\imath}_A \\ (a_y)\,\hat{\jmath}_A \\ (a_z)\,\hat{k}_A \end{array} & \begin{bmatrix} \hat{\imath}_B \cdot \hat{\imath}_A & \hat{\jmath}_B \cdot \hat{\imath}_A & \hat{k}_B \cdot \hat{\imath}_A \\ \hat{\imath}_B \cdot \hat{\jmath}_A & \hat{\jmath}_B \cdot \hat{\jmath}_A & \hat{k}_B \cdot \hat{\jmath}_A \\ \hat{\imath}_B \cdot \hat{k}_A & \hat{\jmath}_B \cdot \hat{k}_A & \hat{k}_B \cdot \hat{k}_A \end{bmatrix} \end{array}$$

Position and Orientation [4]

- Rotation is orthonormal

$$\begin{array}{c|c} {}^A_B R & (b_x)\,\hat{i}_B \quad (b_y)\,\hat{j}_B \quad (b_z)\,\hat{k}_B \\ \hline \begin{array}{c}(a_x)\,\hat{i}_A \\ (a_y)\,\hat{j}_A \\ (a_z)\,\hat{k}_A\end{array} & \begin{bmatrix} \hat{i}_B \cdot \hat{i}_A & \hat{j}_B \cdot \hat{i}_A & \hat{k}_B \cdot \hat{i}_A \\ \hat{i}_B \cdot \hat{j}_A & \hat{j}_B \cdot \hat{j}_A & \hat{k}_B \cdot \hat{j}_A \\ \hat{i}_B \cdot \hat{k}_A & \hat{j}_B \cdot \hat{k}_A & \hat{k}_B \cdot \hat{k}_A \end{bmatrix} \end{array}$$

- The of a rotation matrix inverse = the transpose

$$\mathbf{R} \cdot \mathbf{R}^T = 1$$

→ thus, the ***rows*** are **{A} written in {B}**

$$^B_A\mathbf{R} = {}^A_B\mathbf{R}^T = {}^A_B\mathbf{R}^{-1}$$

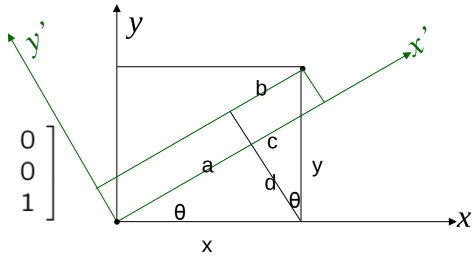
Position and Orientation [5]: A note on orientations

- Orientations, as defined earlier, are represented by three orthonormal vectors
- Only three of these values are unique and we often wish to define a particular rotation using three values (it's easier than specifying 9 orthonormal values)
- There isn't a unique method of specifying the angles that define these transformations

Position and Orientation [6]: "Proof" of Principal Rotation Matrix Terms

- Geometric:

$$\mathbf{R}_z = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$a = x\cos\theta,\ b = y\sin\theta$
$c = y\cos\theta,\ d = x\sin\theta$
Thus:
$x' = x\cos\theta + y\sin\theta$
$y' = -x\sin\theta + y\cos\theta$

Position and Orientation [7]

- Shortcut Notation:

$$\cos(\theta_a) = c\theta_a = \mathbf{c_a}$$
$$\sin(\theta_a) = s\theta_a = \mathbf{s_a}$$

$$\cos(\theta_a + \theta_b) = \mathbf{c_{ab}}$$

$$\therefore \mathbf{s_{ab}} = \boxed{\quad ?}$$

Position and Orientation [8]

- Rotation Formula about the 3 Principal Axes by θ

X:

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Y:

$$\mathbf{R}_y = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

Z:

$$\mathbf{R}_z = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \textit{Homogeneous} \\ \textit{Coordinates} \\ 0 \\ 1 \end{bmatrix}$$

Homogenous Coordinates

$$\hat{p} = \begin{bmatrix} \rho p_x & \rho p_y & \rho p_z & \rho \end{bmatrix}^T$$

- ρ is a scaling value

Homogenous Transformation

$$\begin{bmatrix} {}^{A}R_{B} & {}^{A}p \\ \gamma & \rho \end{bmatrix}$$

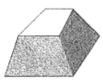
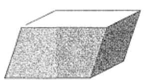
- γ is a projective transformation
- The Homogenous Transformation is a **linear operation** (even if projection is not)

Projective Transformations …

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $\mathbf{l}_\infty$.
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, $\mathbf{I}, \mathbf{J}$ (see section 2.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area

p.44, R. Hartley and A. Zisserman. *Multiple View Geometry in Computer Vision*

Projective Transformations & Other Transformations of 3D Space

Group	Matrix	Distortion	Invariant properties
Projective 15 dof	$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^\mathsf{T} & v \end{bmatrix}$		Intersection and tangency of surfaces in contact. Sign of Gaussian curvature.
Affine 12 dof	$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$		Parallelism of planes, volume ratios, centroids. The plane at infinity, π_∞, (see section 3.5).
Similarity 7 dof	$\begin{bmatrix} s\mathtt{R} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$		The absolute conic, Ω_∞, (see section 3.6).
Euclidean 6 dof	$\begin{bmatrix} \mathtt{R} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$		Volume.

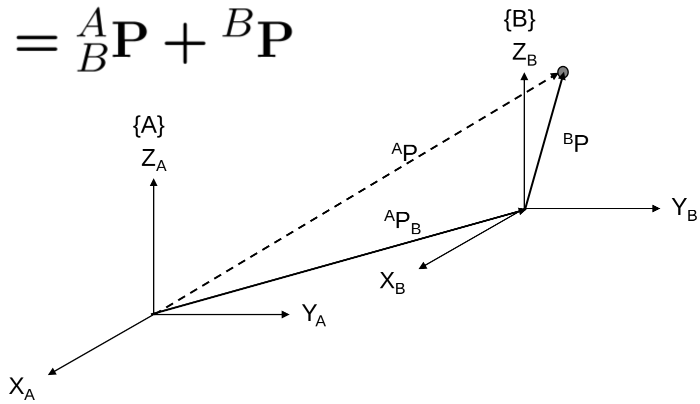
p.78, R. Hartley and A. Zisserman. *Multiple View Geometry in Computer Vision*

Coordinate Transformations [1]

- Translation Again:

  If {B} is translated with respect to {A} **without rotation**, then it is a vector sum

$$ {}^{A}\mathbf{P} = {}^{A}_{B}\mathbf{P} + {}^{B}\mathbf{P} $$



Coordinate Transformations [2]

- Rotation Again:

  {B} is rotated with respect to {A} then
  use rotation matrix to determine new components

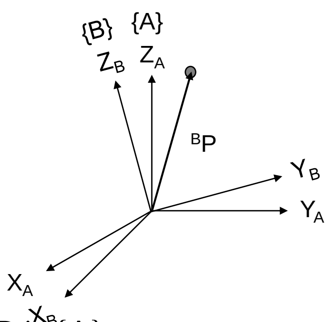
- NOTE:

$$ {}^{A}\mathbf{P} = {}^{A}_{B}\mathbf{R}\,{}^{B}\mathbf{P} $$

  - The Rotation matrix's ***subscript*** matches the position vector's **superscript**

$$ {}^{A}\mathbf{P} = {}^{A}_{[\![B]\!]}\mathbf{R}^{[\![B]\!]}\mathbf{P} $$

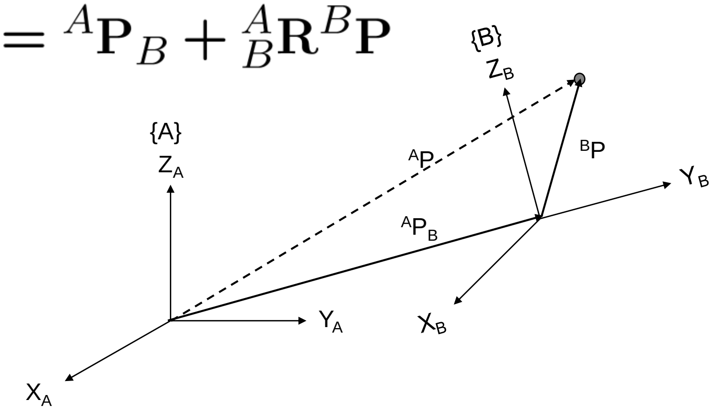
  - This gives Point Positions of {B} ORIENTED in {A}



Coordinate Transformations [3]

- Composite transformation:
  {B} is moved with respect to {A}:

$$^{A}\mathbf{P} = {}^{A}\mathbf{P}_{B} + {}^{A}_{B}\mathbf{R}\,{}^{B}\mathbf{P}$$



General Coordinate Transformations [1]

- A compact representation of the translation and rotation is known as the **Homogeneous Transformation**

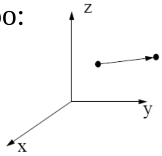
$${}^{A}_{B}\mathbf{T} = \begin{bmatrix} & {}^{A}_{B}\mathbf{R} & & {}^{A}\mathbf{P}_{B} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

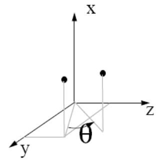
- This allows us to cast the rotation and translation of the general transform in a single matrix form

$$\begin{bmatrix} {}^{A}\mathbf{P} \\ 1 \end{bmatrix} = {}^{A}_{B}\mathbf{T} \begin{bmatrix} {}^{B}\mathbf{P} \\ 1 \end{bmatrix}$$

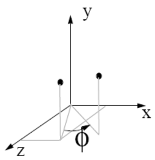
General Coordinate Transformations [2]

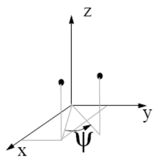
- Similarly, fundamental orthonormal transformations can be represented in this form too:





$$Trans(u, v, w) = \begin{bmatrix} 1 & 0 & 0 & u \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & w \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad Rotx(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\theta & -s\theta & 0 \\ 0 & s\theta & c\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



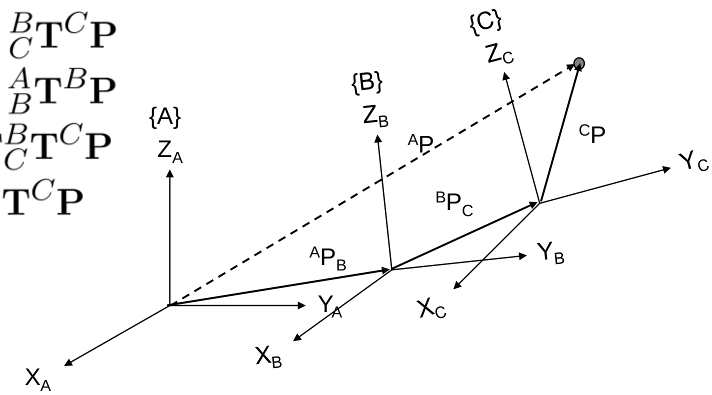


$$Roty(\phi) = \begin{bmatrix} c\phi & 0 & s\phi & 0 \\ 0 & 1 & 0 & 0 \\ -s\phi & 0 & c\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad Rotz(\psi) = \begin{bmatrix} c\psi & -s\psi & 0 & 0 \\ s\psi & c\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

General Coordinate Transformations [3]

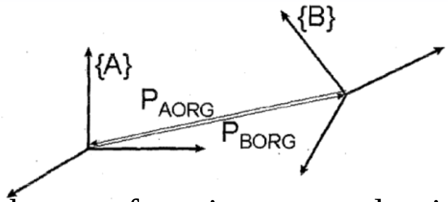
- Multiple transformations compounded as a chain

$$\begin{aligned} {}^{B}\mathbf{P} &= {}^{B}_{C}\mathbf{T}\,{}^{C}\mathbf{P} \\ {}^{A}\mathbf{P} &= {}^{A}_{B}\mathbf{T}\,{}^{B}\mathbf{P} \\ &= {}^{A}_{B}\mathbf{T}\,{}^{B}_{C}\mathbf{T}\,{}^{C}\mathbf{P} \\ &= {}^{A}_{C}\mathbf{T}\,{}^{C}\mathbf{P} \end{aligned}$$



$${}^{A}_{C}\mathbf{T} = \begin{bmatrix} {}^{A}_{B}\mathbf{R}\,{}^{B}_{C}\mathbf{R} & {}^{A}\mathbf{P}_B + {}^{A}_{B}\mathbf{R}\,{}^{B}\mathbf{P}_C \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$

Inverse of a Homogeneous Transformation Matrix



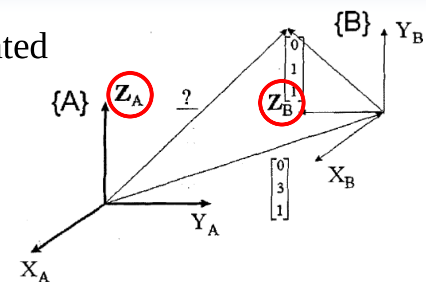
- The inverse of the transform is **not** equal to its transpose because this 4×4 matrix is not orthonormal ($T^{-1} \neq T^T$)
- Invert by parts to give:

$$ {}^{A}_{B}T = \begin{bmatrix} {}^{A}_{B}R & {}^{A}\mathbf{p}_{Borg/O_A} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} $$

$$ {}^{A}_{B}T^{-1} = {}^{B}_{A}T = \begin{bmatrix} {}^{A}_{B}R^T & -{}^{A}_{B}R^T \cdot {}^{A}\mathbf{p}_{Borg/O_A} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^{B}_{A}R & {}^{B}\mathbf{p}_{Aorg/O_B} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} $$

Tutorial Problem

The origin of frame *{B}* is translated to a position [0 3 1] with respect to frame *{A}*.



We would like to find:

1. The homogeneous transformation between the two frames in the figure.
2. For a point *P* defined as as [0 1 1] in frame *{B}*, we would like to find the vector describing this point with respect to frame *{A}*.

Tutorial Solution

• The matrix ${}_BT^A$ is formed as defined earlier:

• Since P in the frame is:

• We find vector **p** in frame *{A}* using the relationship

➔

- The matrix ${}_BT^A$ is formed as defined earlier:

$$ {}^A_BT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

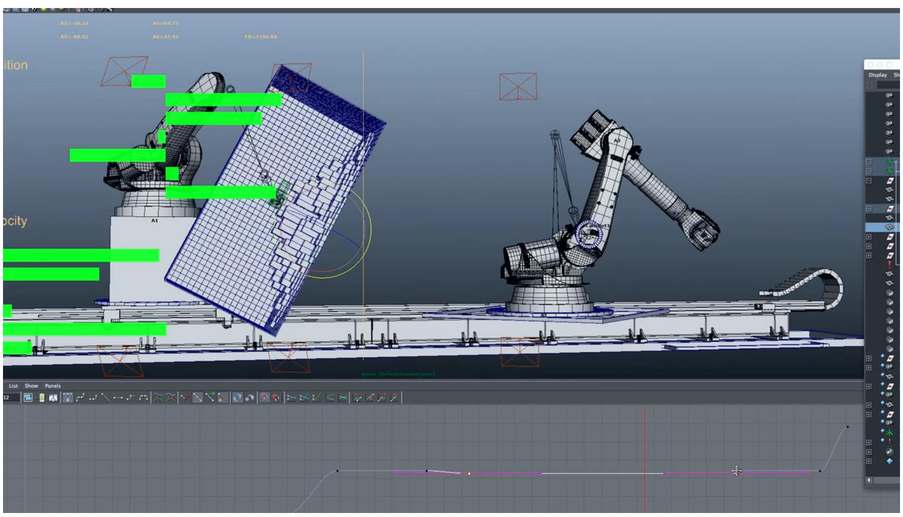
- Since P in the frame is: ${}^B\mathbf{p} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

- We find vector **p** in frame *{A}* using the relationship

$$ {}^Ap = {}^A_BT\,{}^Bp $$

➔ ${}^A\mathbf{p} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix}$

Cool Robotics Share



Forward

Kinematics

→ → → →

Forward Kinematics [1]

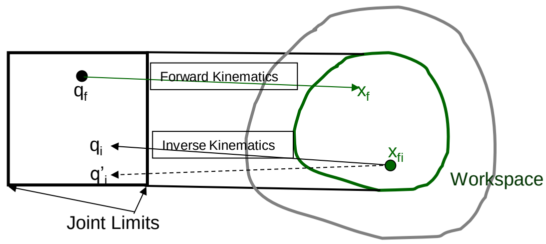
- Forward kinematics is the process of chaining homogeneous transforms together. For example to:
  - Find the articulations of a mechanism, or
  - the fixed transformation between two frames which is known in terms of linear and rotary parameters.
- Calculates the final position from the **machine** (**joint variables**)
- Unique for an open kinematic chain (**serial arm**)
- "Complicated" (multiple solutions, etc.) for a closed kinematic chain (**parallel arm**)

Forward Kinematics [2]

- Can think of this as "spaces":
  - Workspace (x,y,z,α,β,γ): The robot's position & orientation
  - Joint space ($\theta_1 \dots \theta_n$): A state-space vector of joint variables

$$\vec{\mathbf{x}} = \begin{bmatrix} \vec{\mathbf{p}} \\ \vec{\Theta} \end{bmatrix}$$

$$\vec{\mathbf{q}} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

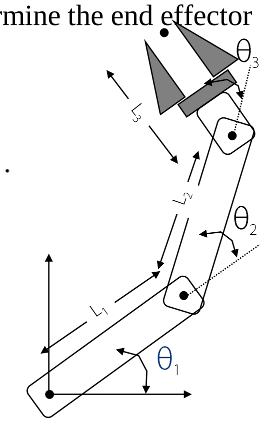


Forward Kinematics [3]

- Consider a planar RRR manipular
- Given the joint angles and link lengths, we can determine the end effector pose:

$$x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + \dots$$
$$L_3 \cos(\theta_1 + \theta_2 + \theta_3)$$

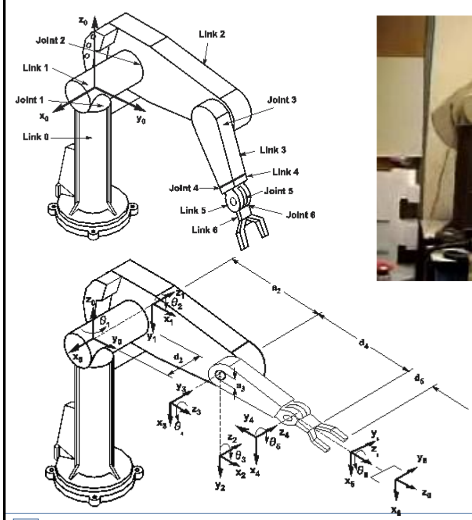
$$y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + \dots$$
$$L_3 \sin(\theta_1 + \theta_2 + \theta_3)$$



- This isn't too difficult to determine for a simple, planar manipulator. BUT …

Forward Kinematics [4]: The PUMA 560!

- What about a more complicated mechanism?



$$\begin{pmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$) - s_1(s_4c_5c_6 + c_4s_6)$

$) + c_1(s_4c_5c_6 + c_4s_6)$

$s_6) - s_1(-s_4c_5s_6 + c_4c_6)$

$s_6) + c_1(-s_4c_5s_6 + c_4c_6)$

$a_x = c_1(c_{23}c_4s$

$a_y = s_1(c_{23}c_4s$

$a_z = -s_{23}c_4s_5$

$p_x = c_1(d_6(c_{23}$

$p_y = s_1(d_6(c_{23}$

$p_z = d_6(c_{23}c_5$

Denavit Hartenberg [DH] Notation

- J. Denavit and R. S. Hartenberg first proposed the use of homogeneous transforms for articulated mechanisms

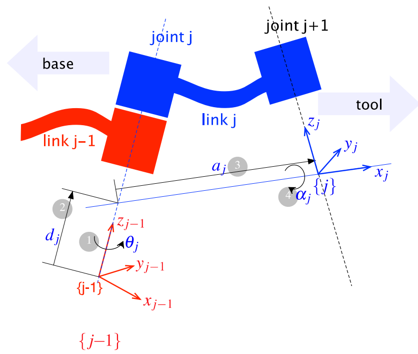
  (But B. Roth, introduced it to robotics)

- A kinematics "short-cut" that reduced the number of parameters by adding a structure to frame selection

- For two frames positioned in space, the first can be moved into coincidence with the second by a sequence of 4 operations:
  - rotate around the x_{i-1} axis by an angle α_i
  - translate along the x_{i-1} axis by a distance a_i
  - translate along the new z axis by a distance d_i
  - rotate around the new z axis by an angle θ_i

Denavit-Hartenberg Convention

- link length a_i the offset distance between the z_{i-1} and z_i axes along the x_i axis;
- link twist α_i the angle from the z_{i-1} axis to the z_i axis about the x_i axis;



Art c/o P. Corke

- link offset d_i the distance from the origin of frame i-1 to the x_i axis along the z_{i-1} axis;
- joint angle θ_i the angle between the x_{i-1} and x_i axes about the z_{i-1} axis.

DH: Where to place frame?

1. Align an axis along principal motion
   1. Rotary (**R**): align rotation axis along the z axis
   2. Prismatic (**P**): align slider travel along x axis
2. Orient so as to position x axis towards next frame
3. θ (rot *z*) → d (trans *z*) → a (trans *x*) → α (rot *x*)

Denavit-Hartenberg → Rotation Matrix

- Each transformation is a product of 4 “basic” transformations (instead of 6)

DH Example [1]: RRR Link Manipulator

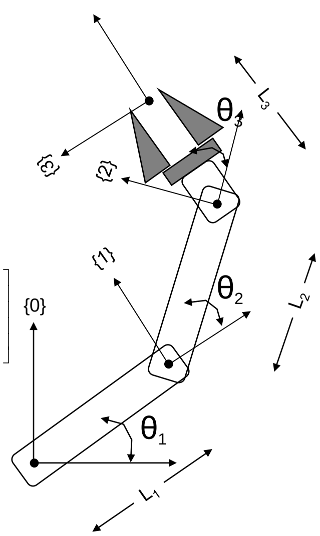
1. Assign the frames at the joints …
2. Fill DH Table …

Link	a_i	α_i	d_i	θ_i
1	L_1	0	0	$\boldsymbol{\theta_1}$
2	L_2	0	0	$\boldsymbol{\theta_2}$
3	L_3	0	0	$\boldsymbol{\theta_3}$

$$^0A_1 = \begin{bmatrix} c_{\theta_1} & -s_{\theta_1} & 0 & L_1 c_{\theta_1} \\ s_{\theta_1} & c_{\theta_1} & 0 & L_1 s_{\theta_1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \; ^1A_2 = \begin{bmatrix} c_{\theta_2} & -s_{\theta_2} & 0 & L_2 c_{\theta_2} \\ s_{\theta_2} & c_{\theta_2} & 0 & L_2 s_{\theta_2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \; ^2A_3 = \begin{bmatrix} c_{\theta_3} & -s_{\theta_3} & 0 & L_3 c_{\theta_3} \\ s_{\theta_3} & c_{\theta_3} & 0 & L_3 s_{\theta_3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0T_3 = {}^0A_1 \, {}^1A_2 \, {}^2A_3$$

$$= \begin{bmatrix} c_{\theta_{123}} & -s_{\theta_{123}} & 0 & L_1 c_{\theta_1} + L_2 c_{\theta_{12}} + L_3 c_{\theta_{123}} \\ s_{\theta_{123}} & c_{\theta_{123}} & 0 & L_1 s_{\theta_1} + L_2 s_{\theta_{12}} + L_3 s_{\theta_{123}} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



DH Example [2]: RRP Link Manipulator

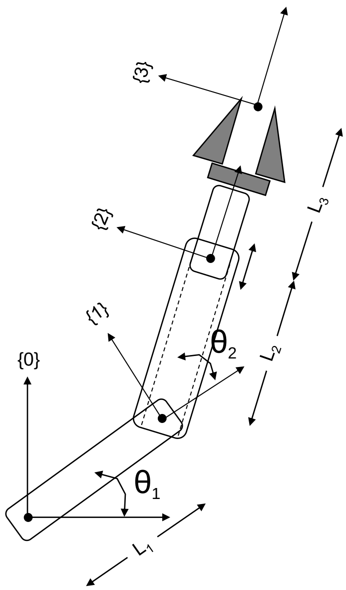
1. Assign the frames at the joints …
2. Fill DH Table …

Link	a_i	α_i	d_i	θ_i
1	L_1	0	0	$\boldsymbol{\theta_1}$
2	L_2	0	0	$\boldsymbol{\theta_2}$
3	$\mathbf{L_3}$	0	0	0

$$ {}^0A_1 = \begin{bmatrix} c_{\theta_1} & -s_{\theta_1} & 0 & L_1 c_{\theta_1} \\ s_{\theta_1} & c_{\theta_1} & 0 & L_1 s_{\theta_1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^1A_2 = \begin{bmatrix} c_{\theta_2} & -s_{\theta_2} & 0 & L_2 c_{\theta_2} \\ s_{\theta_2} & c_{\theta_2} & 0 & L_2 s_{\theta_2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^2A_3 = \begin{bmatrix} 1 & 0 & 0 & L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0T_3 = {}^0A_1\,{}^1A_2\,{}^2A_3 $$

$$ = \begin{bmatrix} c_{\theta_{12}} & -s_{\theta_{12}} & 0 & L_1 c_{\theta_1} + (L_2 + L_3) c_{\theta_{12}} \\ s_{\theta_{12}} & c_{\theta_{12}} & 0 & L_1 s_{\theta_1} + (L_2 + L_3) s_{\theta_{12}} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



DH Example [3]: Puma 560

- "Simple" 6R robot exercise for the reader …

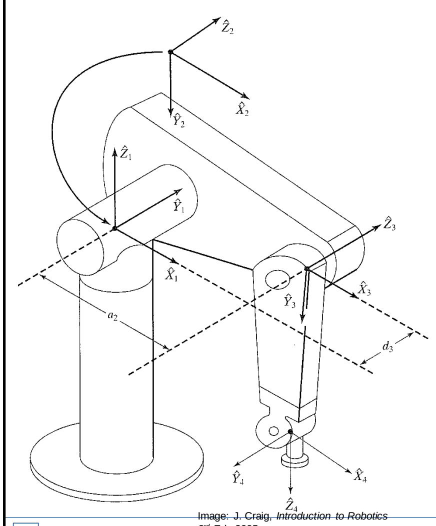
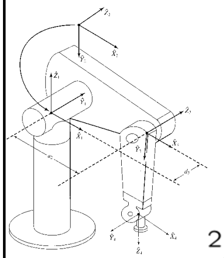


Image: J. Craig, *Introduction to Robotics* 3rd Ed., 2005

Link	a_i	α_i	d_i	θ_i
1	0	0	0	$\boldsymbol{\theta_1}$
2	0	$-\pi/2$	0	$\boldsymbol{\theta_2}$
3	L_2	0	D_3	$\boldsymbol{\theta_3}$
4	L_3	$-\pi/2$	D_4	$\boldsymbol{\theta_4}$
5	0	$\pi/2$	0	$\boldsymbol{\theta_5}$
6	0	$-\pi/2$	0	$\boldsymbol{\theta_6}$

DH Example [3]: Puma 560 [2]



$$^0A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ^1A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ -s_2 & -c_2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

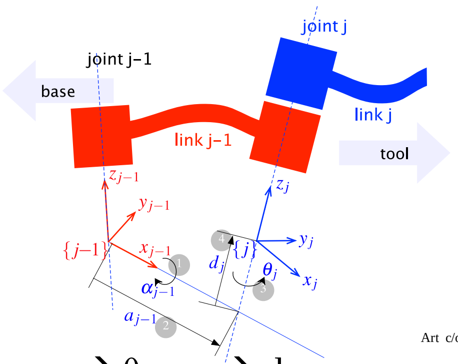
$$^2A_3 = \begin{bmatrix} c_3 & -s_3 & 0 & L_2 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ^3A_4 = \begin{bmatrix} c_4 & -s_4 & 0 & L_3 \\ 0 & 0 & 1 & d_4 \\ -s_4 & -c_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^4A_5 = \begin{bmatrix} c_4 & -s_5 & 0 & L_3 \\ 0 & 0 & 1 & d_4 \\ -s_5 & -c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ^5A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & L_3 \\ 0 & 0 & -1 & 0 \\ -s_6 & -c_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0T_6 = {}^0A_1{}^1A_2{}^2A_3{}^3A_4{}^4A_5{}^5A_6$$

Modified DH

- Made "popular" by Craig's *Intro. to Robotics* book
- Link coordinates attached to the near by joint

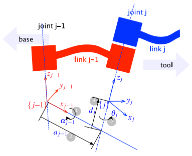


Art c/o P. Corke

- a (trans *x-1*) → α (rot *x-1*) → θ (rot *z*) → d (trans *z*)

Modified DH [2]

- Gives a similar result
  (but it's not commutative)



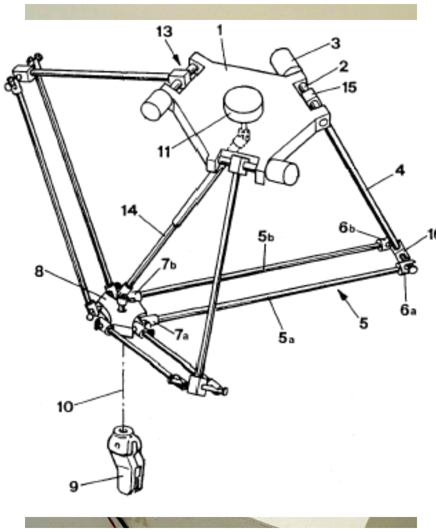
$$\Rightarrow {}^{i-1}A_i = R_x(\alpha_{i-1})\, T_x(a_{i-1})\, R_z(\theta_i)\, T_x(d_i)$$

- Refactoring Standard → to Modified

$$\underbrace{\{R_z(\theta_1) T_z(d_1) T_x(a_1) R_x(\alpha_1)\}}_{\text{DH}_1} \cdot \underbrace{\{R_z(\theta_2) T_z(d_2) T_x(a_2) R_x(\alpha_2)\}}_{\text{DH}_2} \cdot \underbrace{\{R_z(\theta_3) T_z(d_3)\}}_{\text{End Effector}}$$

$$= \underbrace{\{R_z(\theta_1) T_z(d_1)\}}_{\text{Base}} \cdot \underbrace{\{T_x(a_1) R_x(\alpha_1) R_z(\theta_2) T_z(d_2)\}}_{\text{MDH}_1} \cdot \underbrace{\{T_x(a_2) R_x(\alpha_2) R_z(\theta_3) T_z(d_3)\}}_{\text{MDH}_2}$$

Parallel Manipulators



Sources: Wikipedia, "Delta Robot", ParallelMic.Org, "Delta Parallel Robot", and US Patent 4,976,582

- The "central" Kinematic structure is made up of closed-loop chain(s)
- Compared to Serial Mechanisms:
  + Higher Stiffness
  + Higher Payload
  + Less Inertia
  – Smaller Workspace
  – Coordinated Drive System
  – More Complex & $$$

Symmetrical Parallel Manipulator

A sub-class of Parallel Manipulator:

- # Limbs (***m***) = # DOF (***F***)
- The joints are arranged in an identical pattern
- The # and location of actuated joints are the same

Thus:

- Number of Loops (L): One less than # of limbs

$$L = m - 1 = F - 1$$

- Connectivity (C_k)

$$\sum_{k=1}^{m} C_k = (\lambda + 1)\, F - \lambda$$

Where: λ: The DOF of the space that the system is in (e.g., λ=6 for 3D space).

Cool Robotics Share

http://www.kinemasystems.com/