METR4202 -- Robotics
Tutorial 2 – Week 2: Homogeneous Coordinates

The objective of this tutorial is to explore homogenous transformations. The MATLAB robotics toolbox developed by Peter Corke might be a useful aid[1].

Please answer the tutorial by Thursday night via the Platypus system for tutor/peer feedback.

Reading

Please read/review Section 2.4 of *Multiple View Geometry in Computer Vision* (see attached).
(from R. Hartley and A. Zisserman. *Multiple View Geometry in Computer Vision*. Cambridge University Press, 2004)

Review

The Homogeneous Transformations functions in the toolbox are useful.

1. Try the Transformations module of `rtbdemo` for a demonstration of these function
2. Look at rpy2tr and tr2rpy (`doc rpy2tr` and `doc tr2rpy`)
3. Look at the source of these functions (`open rpy2tr` and `open tr2rpy`). Does `tr2rpy` exploit the redundancies inherent in a rotation matrix?

Questions

1. Calculate the homogeneous transformation matrix ${}^{A}_{B}T$ given the translations (${}^{A}\boldsymbol{P}_{B}$) and the roll-pitch-yaw rotations (as α-β-γ) applied in the order yaw, pitch, roll. **[20%]**

   a. $\alpha=10°$, $\beta=20°$, $\gamma=30°$, ${}^{A}\boldsymbol{P}_{B}=\{1\ \ 2\ \ 3\}^{\mathbf{T}}$
   b. $\alpha=10°$, $\beta=30°$, $\gamma=30°$, ${}^{A}\boldsymbol{P}_{B}=\{3\ \ 0\ \ 0\}^{\mathbf{T}}$

2. Compare the output of: $\alpha=90°$, $\beta=180°$, $\gamma=-90°$, ${}^{A}\boldsymbol{P}_{B}=\{0\ \ 0\ \ 1\}^{\mathbf{T}}$ and $\alpha=90°$, $\beta=180°$, $\gamma=270°$, ${}^{A}\boldsymbol{P}_{B}=\{0\ \ 0\ \ 1\}^{\mathbf{T}}$ **[10%]**

3. Given the following 3x3 rotation matrices: **[40%]**

$$R_1 = \begin{bmatrix} 0.7500 & -0.4330 & -0.5000 \\ 0.2165 & 0.8750 & -0.4330 \\ 0.6250 & 0.2165 & 0.7500 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0.6399 & -0.2351 & -0.6159 \\ 0.2860 & 0.5854 & -0.4970 \\ 0.3221 & 0.2488 & 0.7132 \end{bmatrix},$$

$$R_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0.8660 & 0.5000 & 0 \\ -0.500 & 0.8660 & 0 \end{bmatrix}, \quad R_4 = \begin{bmatrix} 0.0238 & 0.1524 & 0.9880 \\ -0.3030 & -0.9407 & 0.1524 \\ 0.9527 & -0.3030 & 0.0238 \end{bmatrix}$$

   a. Are these (within practical numerical limits) valid rotation matrices? Why?

   b. If yes, determine the Roll, Pitch, and Yaw that define each matrix. Do you believe their values?

4. Thought Experiment: **[Extra Time – 30%]**

Review the even and odd nature of sine/cosine functions. Using this, explain how many positive and negative terms you should expect in a rotation matrix after an Euler Angle triad (e.g., Z-Y-X moving).

[1] http://petercorke.com/Robotics_Toolbox.html

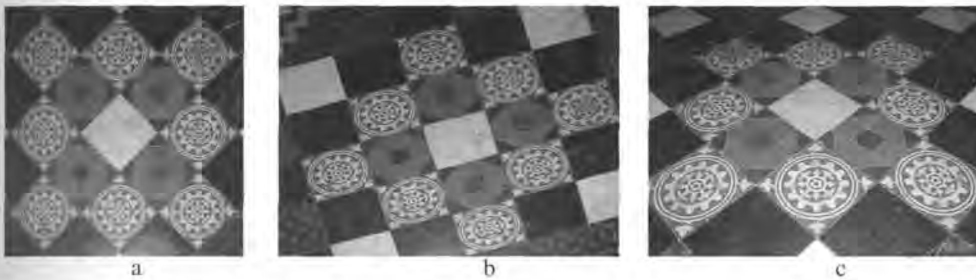


Fig. 2.6. **Distortions arising under central projection.** *Images of a tiled floor. (a)* **Similarity***: the circular pattern is imaged as a circle. A square tile is imaged as a square. Lines which are parallel or perpendicular have the same relative orientation in the image. (b)* **Affine***: The circle is imaged as an ellipse. Orthogonal world lines are not imaged as orthogonal lines. However, the sides of the square tiles, which are parallel in the world are parallel in the image. (c)* **Projective***: Parallel world lines are imaged as converging lines. Tiles closer to the camera have a larger image than those further away.*

which is a quadratic form $\mathbf{x}'^{\mathsf{T}}\mathtt{C}'\mathbf{x}'$ with $\mathtt{C}' = \mathtt{H}^{-\mathsf{T}}\mathtt{C}\mathtt{H}^{-1}$. This gives the transformation rule for a conic:

Result 2.13. *Under a point transformation* $\mathbf{x}' = \mathtt{H}\mathbf{x}$, *a conic* $\mathtt{C}$ *transforms to* $\mathtt{C}' = \mathtt{H}^{-\mathsf{T}}\mathtt{C}\mathtt{H}^{-1}$.

The presence of $\mathtt{H}^{-1}$ in this equation may be expressed by saying that a conic transforms *covariantly*. The transformation rule for a dual conic is derived in a similar manner. This gives:

Result 2.14. *Under a point transformation* $\mathbf{x}' = \mathtt{H}\mathbf{x}$, *a dual conic* $\mathtt{C}^*$ *transforms to* $\mathtt{C}^{*\prime} = \mathtt{H}\mathtt{C}^*\mathtt{H}^{\mathsf{T}}$.

2.4 A hierarchy of transformations

In this section we describe the important specializations of a projective transformation and their geometric properties. It was shown in section 2.3 that projective transformations form a group. This group is called the *projective linear group*, and it will be seen that these specializations are *subgroups* of this group.

The group of invertible $n \times n$ matrices with real elements is the (real) general linear group on n dimensions, or $GL(n)$. To obtain the projective linear group the matrices related by a scalar multiplier are identified, giving $PL(n)$ (this is a quotient group of $GL(n)$). In the case of projective transformations of the plane $n = 3$.

The important subgroups of $PL(3)$ include the *affine group*, which is the subgroup of $PL(3)$ consisting of matrices for which the last row is $(0, 0, 1)$, and the *Euclidean group*, which is a subgroup of the affine group for which in addition the upper left hand 2×2 matrix is orthogonal. One may also identify the *oriented Euclidean group* in which the upper left hand 2×2 matrix has determinant 1.

We will introduce these transformations starting from the most specialized, the isometries, and progressively generalizing until projective transformations are reached.

This defines a *hierarchy* of transformations. The distortion effects of various transformations in this hierarchy are shown in figure 2.6.

Some transformations of interest are not groups, for example, perspectivities (because the composition of two perspectivities is a projectivity, not a perspectivity). This point is covered in section A7.4(p632).

Invariants. An alternative to describing the transformation *algebraically*, i.e. as a matrix acting on coordinates of a point or curve, is to describe the transformation in terms of those elements or quantities that are preserved or *invariant*. A (scalar) invariant of a geometric configuration is a function of the configuration whose value is unchanged by a particular transformation. For example, the separation of two points is unchanged by a Euclidean transformation (translation and rotation), but not by a similarity (e.g. translation, rotation and isotropic scaling). Distance is thus a Euclidean, but not similarity invariant. The angle between two lines is both a Euclidean and a similarity invariant.

2.4.1 Class I: Isometries

Isometries are transformations of the plane $\mathbb{R}^2$ that preserve Euclidean distance (from *iso* = same, *metric* = measure). An isometry is represented as

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \epsilon\cos\theta & -\sin\theta & t_x \\ \epsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

where $\epsilon = \pm 1$. If $\epsilon = 1$ then the isometry is *orientation-preserving* and is a *Euclidean* transformation (a composition of a translation and rotation). If $\epsilon = -1$ then the isometry reverses orientation. An example is the composition of a reflection, represented by the matrix $\text{diag}(-1, 1, 1)$, with a Euclidean transformation.

Euclidean transformations model the motion of a rigid object. They are by far the most important isometries in practice, and we will concentrate on these. However, the orientation reversing isometries often arise as ambiguities in structure recovery.

A planar Euclidean transformation can be written more concisely in block form as

$$\mathbf{x}' = \mathtt{H}_{\text{E}}\mathbf{x} = \begin{bmatrix} \mathtt{R} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix} \mathbf{x} \qquad (2.7)$$

where $\mathtt{R}$ is a 2×2 rotation matrix (an orthogonal matrix such that $\mathtt{R}^{\mathsf{T}}\mathtt{R} = \mathtt{R}\mathtt{R}^{\mathsf{T}} = \mathtt{I}$), $\mathbf{t}$ a translation 2-vector, and $\mathbf{0}$ a null 2-vector. Special cases are a pure rotation (when $\mathbf{t} = \mathbf{0}$) and a pure translation (when $\mathtt{R} = \mathtt{I}$). A Euclidean transformation is also known as a *displacement*.

A planar Euclidean transformation has three degrees of freedom, one for the rotation and two for the translation. Thus three parameters must be specified in order to define the transformation. The transformation can be computed from two point correspondences.

Invariants. The invariants are very familiar, for instance: length (the distance between two points), angle (the angle between two lines), and area.

Groups and orientation. An isometry is orientation-preserving if the upper left hand 2×2 matrix has determinant 1. Orientation-*preserving* isometries form a group, orientation-*reversing* ones do not. This distinction applies also in the case of similarity and affine transformations which now follow.

2.4.2 Class II: Similarity transformations

A similarity transformation (or more simply a *similarity*) is an isometry composed with an isotropic scaling. In the case of a Euclidean transformation composed with a scaling (i.e. no reflection) the similarity has matrix representation

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}. \tag{2.8}$$

This can be written more concisely in block form as

$$\mathbf{x}' = \mathtt{H}_S\mathbf{x} = \begin{bmatrix} s\mathtt{R} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \mathbf{x} \tag{2.9}$$

where the scalar s represents the isotropic scaling. A similarity transformation is also known as an *equi-form* transformation, because it preserves "shape" (form). A planar similarity transformation has four degrees of freedom, the scaling accounting for one more degree of freedom than a Euclidean transformation. A similarity can be computed from two point correspondences.

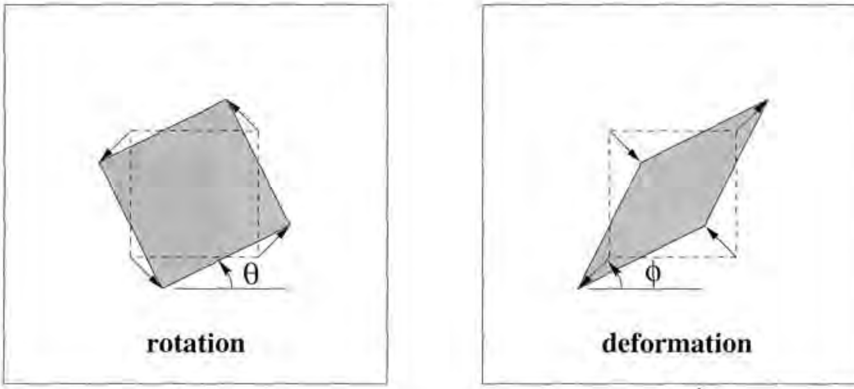
Invariants. The invariants can be constructed from Euclidean invariants with suitable provision being made for the additional scaling degree of freedom. Angles between lines are not affected by rotation, translation or isotropic scaling, and so are similarity invariants. In particular parallel lines are mapped to parallel lines. The length between two points is not a similarity invariant, but the *ratio* of two lengths is an invariant, because the scaling of the lengths cancels out. Similarly a ratio of areas is an invariant because the scaling (squared) cancels out.

Metric structure. A term that will be used frequently in the discussion on reconstruction (chapter 10) is *metric*. The description *metric structure* implies that the structure is defined up to a similarity.

2.4.3 Class III: Affine transformations

An affine transformation (or more simply an *affinity*) is a non-singular linear transformation followed by a translation. It has the matrix representation

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{2.10}$$



a b

Fig. 2.7. **Distortions arising from a planar affine transformation.** *(a) Rotation by* $\mathtt{R}(\theta)$. *(b) A deformation* $\mathtt{R}(-\phi)\,\mathtt{D}\,\mathtt{R}(\phi)$. *Note, the scaling directions in the deformation are orthogonal.*

or in block form

$$\mathbf{x}' = \mathtt{H}_{\mathtt{A}}\mathbf{x} = \begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix} \mathbf{x} \tag{2.11}$$

with $\mathtt{A}$ a 2×2 non-singular matrix. A planar affine transformation has six degrees of freedom corresponding to the six matrix elements. The transformation can be computed from three point correspondences.

A helpful way to understand the geometric effects of the linear component $\mathtt{A}$ of an affine transformation is as the composition of two fundamental transformations, namely rotations and non-isotropic scalings. The affine matrix $\mathtt{A}$ can always be decomposed as

$$\mathtt{A} \;=\; \mathtt{R}(\theta)\,\mathtt{R}(-\phi)\,\mathtt{D}\,\mathtt{R}(\phi) \tag{2.12}$$

where $\mathtt{R}(\theta)$ and $\mathtt{R}(\phi)$ are rotations by θ and ϕ respectively, and $\mathtt{D}$ is a diagonal matrix:

$$\mathtt{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

This decomposition follows directly from the SVD (section A4.4(p585)): writing $\mathtt{A} = \mathtt{UDV}^{\mathsf{T}} = (\mathtt{UV}^{\mathsf{T}})(\mathtt{VDV}^{\mathsf{T}}) = \mathtt{R}(\theta)\,(\mathtt{R}(-\phi)\,\mathtt{D}\,\mathtt{R}(\phi))$, since $\mathtt{U}$ and $\mathtt{V}$ are orthogonal matrices.

The affine matrix $\mathtt{A}$ is hence seen to be the concatenation of a rotation (by ϕ); a scaling by λ_1 and λ_2 respectively in the (rotated) x and y directions; a rotation back (by $-\phi$); and finally another rotation (by θ). The only "new" geometry, compared to a similarity, is the non-isotropic scaling. This accounts for the two extra degrees of freedom possessed by an affinity over a similarity. They are the angle ϕ specifying the scaling direction, and the ratio of the scaling parameters $\lambda_1 : \lambda_2$. The essence of an affinity is this scaling in orthogonal directions, oriented at a particular angle. Schematic examples are given in figure 2.7.

Invariants. Because an affine transformation includes non-isotropic scaling, the similarity invariants of length ratios and angles between lines are not preserved under an affinity. Three important invariants are:

(i) **Parallel lines.** Consider two parallel lines. These intersect at a point $(x_1, x_2, 0)^\top$ at infinity. Under an affine transformation this point is mapped to another point at infinity. Consequently, the parallel lines are mapped to lines which still intersect at infinity, and so are parallel after the transformation.

(ii) **Ratio of lengths of parallel line segments.** The length scaling of a line segment depends only on the angle between the line direction and scaling directions. Suppose the line is at angle α to the x-axis of the orthogonal scaling direction, then the scaling magnitude is $\sqrt{\lambda_1^2 \cos^2 \alpha + \lambda_2^2 \sin^2 \alpha}$. This scaling is common to all lines with the same direction, and so cancels out in a ratio of parallel segment lengths.

(iii) **Ratio of areas.** This invariance can be deduced directly from the decomposition (2.12). Rotations and translations do not affect area, so only the scalings by λ_1 and λ_2 matter here. The effect is that area is scaled by $\lambda_1 \lambda_2$ which is equal to $\det \mathtt{A}$. Thus the area of any shape is scaled by $\det \mathtt{A}$, and so the scaling cancels out for a ratio of areas. It will be seen that this does not hold for a projective transformation.

An affinity is orientation-preserving or -reversing according to whether $\det \mathtt{A}$ is positive or negative respectively. Since $\det \mathtt{A} = \lambda_1 \lambda_2$ the property depends only on the sign of the scalings.

2.4.4 Class IV: Projective transformations

A projective transformation was defined in (2.5). It is a general non-singular linear transformation of *homogeneous* coordinates. This generalizes an affine transformation, which is the composition of a general non-singular linear transformation of *inhomogeneous* coordinates and a translation. We have earlier seen the action of a projective transformation (in section 2.3). Here we examine its block form

$$\mathbf{x}' = \mathtt{H}_P \mathbf{x} = \begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^\top & v \end{bmatrix} \mathbf{x} \qquad (2.13)$$

where the vector $\mathbf{v} = (v_1, v_2)^\top$. The matrix has nine elements with only their ratio significant, so the transformation is specified by eight parameters. Note, it is not always possible to scale the matrix such that v is unity since v might be zero. A projective transformation between two planes can be computed from four point correspondences, with no three collinear on either plane. See figure 2.4.

Unlike the case of affinities, it is not possible to distinguish between orientation preserving and orientation reversing projectivities in $\mathbb{P}^2$. We will return to this point in section 2.6.

Invariants. The most fundamental projective invariant is the cross ratio of four collinear points: a ratio of lengths on a line is invariant under affinities, but not under projectivities. However, a ratio of ratios or *cross ratio* of lengths on a line is a projective invariant. We return to properties of this invariant in section 2.5.

2.4.5 Summary and comparison

Affinities (6 dof) occupy the middle ground between similarities (4 dof) and projectivities (8 dof). They generalize similarities in that angles are not preserved, so that shapes are skewed under the transformation. On the other hand their action is homogeneous over the plane: for a given affinity the $\det \mathtt{A}$ scaling in area of an object (e.g. a square) is the same anywhere on the plane; and the orientation of a transformed line depends only on its initial orientation, not on its position on the plane. In contrast, for a given projective transformation, area scaling varies with position (e.g. under perspective a more distant square on the plane has a smaller image than one that is nearer, as in figure 2.6); and the orientation of a transformed line depends on both the orientation and position of the source line (however, it will be seen later in section 8.6(p213) that a line's vanishing point depends only on line orientation, not position).

The key difference between a projective and affine transformation is that the vector $\mathbf{v}$ is not null for a projectivity. This is responsible for the non-linear effects of the projectivity. Compare the mapping of an ideal point $(x_1, x_2, 0)^\mathsf{T}$ under an affinity and projectivity: First the affine transformation

$$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathtt{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ 0 \end{pmatrix}. \tag{2.14}$$

Second the projective transformation

$$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^\mathsf{T} & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathtt{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ v_1 x_1 + v_2 x_2 \end{pmatrix}. \tag{2.15}$$

In the first case the ideal point remains ideal (i.e. at infinity). In the second it is mapped to a finite point. It is this ability which allows a projective transformation to model vanishing points.

2.4.6 Decomposition of a projective transformation

A projective transformation can be decomposed into a chain of transformations, where each matrix in the chain represents a transformation higher in the hierarchy than the previous one.

$$\mathtt{H} = \mathtt{H}_\mathtt{S}\,\mathtt{H}_\mathtt{A}\,\mathtt{H}_\mathtt{P} = \begin{bmatrix} s\mathtt{R} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathtt{K} & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathtt{I} & \mathbf{0} \\ \mathbf{v}^\mathsf{T} & v \end{bmatrix} = \begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^\mathsf{T} & v \end{bmatrix} \tag{2.16}$$

with $\mathtt{A}$ a non-singular matrix given by $\mathtt{A} = s\mathtt{R}\mathtt{K} + \mathbf{t}\mathbf{v}^\mathsf{T}$, and $\mathtt{K}$ an upper-triangular matrix normalized as $\det \mathtt{K} = 1$. This decomposition is valid provided $v \neq 0$, and is unique if s is chosen positive.

Each of the matrices $\mathtt{H}_S, \mathtt{H}_A, \mathtt{H}_P$ is the "essence" of a transformation of that type (as indicated by the subscripts S, A, P). Consider the process of rectifying the perspective image of a plane as in example 2.12: $\mathtt{H}_P$ (2 dof) moves the line at infinity; $\mathtt{H}_A$ (2 dof) affects the affine properties, but does not move the line at infinity; and finally, $\mathtt{H}_S$ is a general similarity transformation (4 dof) which does not affect the affine or projective properties. The transformation $\mathtt{H}_P$ is an *elation*, described in section A7.3(*p*631).

Example 2.15. The projective transformation

$$\mathtt{H} = \begin{bmatrix} 1.707 & 0.586 & 1.0 \\ 2.707 & 8.242 & 2.0 \\ 1.0 & 2.0 & 1.0 \end{bmatrix}$$

may be decomposed as

$$\mathtt{H} = \begin{bmatrix} 2\cos 45^\circ & -2\sin 45^\circ & 1 \\ 2\sin 45^\circ & 2\cos 45^\circ & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}.$$

△

This decomposition can be employed when the objective is to only partially determine the transformation. For example, if one wants to measure length ratios from the perspective image of a plane, then it is only necessary to determine (rectify) the transformation up to a similarity. We return to this approach in section 2.7.

Taking the inverse of $\mathtt{H}$ in (2.16) gives $\mathtt{H}^{-1} = \mathtt{H}_P^{-1}\mathtt{H}_A^{-1}\mathtt{H}_S^{-1}$. Since $\mathtt{H}_P^{-1}$, $\mathtt{H}_A^{-1}$ and $\mathtt{H}_S^{-1}$ are still projective, affine and similarity transformations respectively, a general projective transformation may also be decomposed in the form

$$\mathtt{H} = \mathtt{H}_P\,\mathtt{H}_A\,\mathtt{H}_S = \begin{bmatrix} \mathtt{I} & \mathbf{0} \\ \mathbf{v}^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathtt{K} & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} s\mathtt{R} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \tag{2.17}$$

Note that the actual values of $\mathtt{K}$, $\mathtt{R}$, $\mathbf{t}$ and $\mathbf{v}$ will be different from those of (2.16).

2.4.7 The number of invariants

The question naturally arises as to how many invariants there are for a given geometric configuration under a particular transformation. First the term "number" needs to be made more precise, for if a quantity is invariant, such as length under Euclidean transformations, then any function of that quantity is invariant. Consequently, we seek a counting argument for the number of functionally independent invariants. By considering the number of transformation parameters that must be eliminated in order to form an invariant, it can be seen that:

Result 2.16. *The number of functionally independent invariants is equal to, or greater than, the number of degrees of freedom of the configuration less the number of degrees of freedom of the transformation.*

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $\mathbf{l}_\infty$.
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, **I**, **J** (see section 2.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area

Table 2.1. **Geometric properties invariant to commonly occurring *planar* transformations.** *The matrix* $\mathtt{A} = [a_{ij}]$ *is an invertible* 2×2 *matrix,* $\mathtt{R} = [r_{ij}]$ *is a 2D rotation matrix, and* (t_x, t_y) *a 2D translation. The distortion column shows typical effects of the transformations on a square. Transformations higher in the table can produce all the actions of the ones below. These range from Euclidean, where only translations and rotations occur, to projective where the square can be transformed to any arbitrary quadrilateral (provided no three points are collinear).*

For example, a configuration of four points in general position has 8 degrees of freedom (2 for each point), and so 4 similarity, 2 affinity and zero projective invariants since these transformations have respectively 4, 6 and 8 degrees of freedom.

Table 2.1 summarizes the 2D transformation groups and their invariant properties. Transformations lower in the table are specializations of those above. A transformation lower in the table inherits the invariants of those above.

2.5 The projective geometry of 1D

The development of the projective geometry of a line, $\mathbb{P}^1$, proceeds in much the same way as that of the plane. A point x on the line is represented by homogeneous coordinates $(x_1, x_2)^\mathsf{T}$, and a point for which $x_2 = 0$ is an ideal point of the line. We will use the notation $\bar{\mathbf{x}}$ to represent the 2-vector $(x_1, x_2)^\mathsf{T}$. A projective transformation of a line is represented by a 2×2 homogeneous matrix,

$$\bar{\mathbf{x}}' = \mathtt{H}_{2\times 2}\bar{\mathbf{x}}$$

and has 3 degrees of freedom corresponding to the four elements of the matrix less one for overall scaling. A projective transformation of a line may be determined from three corresponding points.