Inverse Kinematics & Kinetics

METR 4202: **Robotics** & Automation

Dr Surya Singh -- Lecture # 4

August 17, 2016

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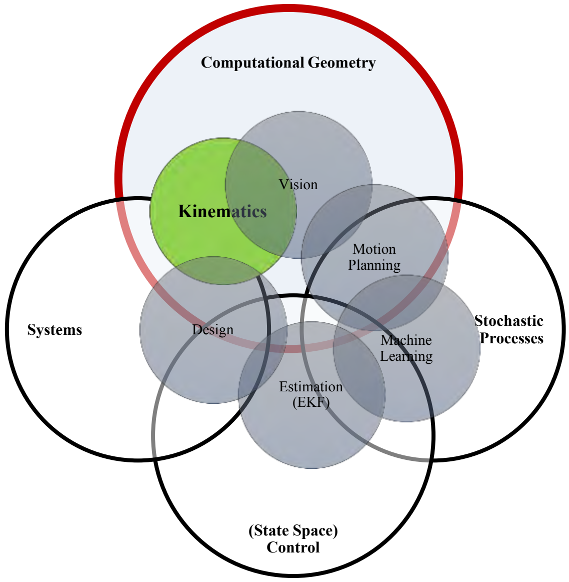
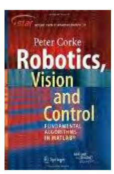
Schedule of Events

Week	Date	Lecture (W: 12:05-1:50, 50-N202)
1	27-Jul	Introduction
2	3-Aug	Representing Position & Orientation & State (Frames, Transformation Matrices & Affine Transformations)
3	10-Aug	Robot Kinematics Review (*& Ekka Day*)
4	**17-Aug**	**Robot Inverse Kinematics & Kinetics**
5	24-Aug	Robot Dynamics (Jacobeans)
6	31-Aug	Robot Sensing: Perception & Linear Observers
7	7-Sep	Robot Sensing: Multiple View Geometry & Feature Detection
8	14-Sep	Probabilistic Robotics: Localization
9	21-Sep	Probabilistic Robotics: SLAM
	28-Sep	*Study break*
10	5-Oct	Motion Planning
11	12-Oct	State-Space Modelling
12	19-Oct	Shaping the Dynamic Response
13	26-Oct	LQR + Course Review

Course Organization



Follow Along Reading:



[Robotics, Vision & Control]
by [Peter Corke]

Also online:[*SpringerLink*]

[UQ Library eBook: 364220144X]

Today

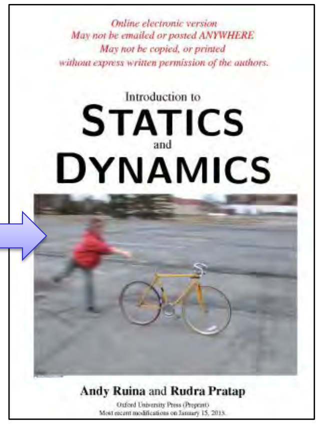
➔ Representing Space ➔

- RVC
  - Chapter 7: Robot Arm Kinematics
  -

- Inverse Kinematics
  - RVC
    §7.3: Robot Arm Kinematics

Next Time

Reference Material



Online:
http://ruina.tam.cornell.edu/Book/RuinaPratap1-15-13.pdf

Inverse Kinematics

Inverse Kinematics [More Generally]

- Freudenstein (1973) referred to the inverse kinematics problem of the most general **6R** manipulator as the "Mount Everest" of kinematic problems.

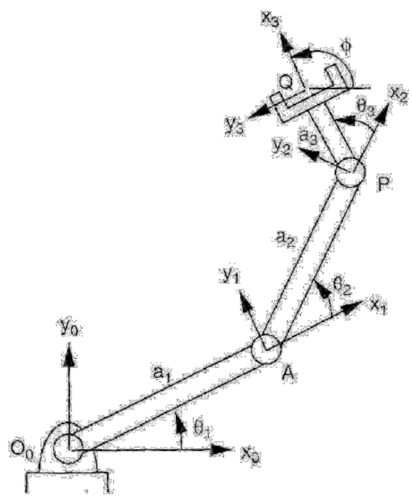
- Tsai and Morgan (1985) and Primrose (1986) proved that this has at most 16 real solutions.

- Duffy and Crane (1980) derived a closed-form solution for the general **7R** single-loop spatial mechanism.
  - The solution was obtained in the form of a 16 x 16 delerminant in which every element is a second-degree polynomial in one joint variable. The determinant, when expended, should yield a 32nd-degree polynomial equation and hence confirms the upper limit predicted by Roth *et al*. (1973).

- Tsai and Morgan (1985) used the homotopy continuation method to solve the inverse kinematics of the general 6R manipulator and found only 16 solutions

- Raghavan and Roth (1989, 1990) used the dyalitic elimination method to derive a 16th-degree polynomial for the general 6R inverse kinematics problem.

Example: FK/IK of a 3R Planar Arm



- Derived from Tsai (p. 63)

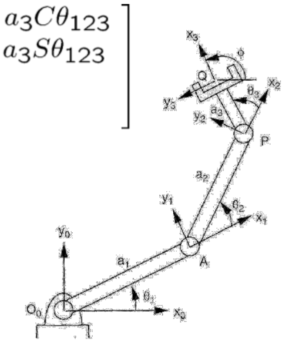
Example: 3R Planar Arm [2]

Position Analysis: 3·Planar 1-R Arm rotating about **Z** **[Ⓩ]**

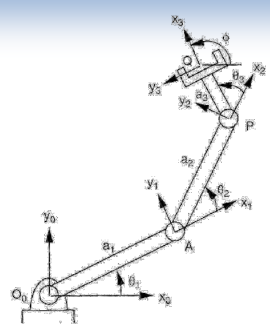
$$ {}^{0}A_3 = {}^{0}A_1 \cdot {}^{1}A_2 \cdot {}^{2}A_3 $$

Substituting gives:

$$ {}^{0}A_3 = \begin{bmatrix} C\theta_{123} & -S\theta_{123} & 0 & a_1C\theta_1 + a_2C\theta_{12} + a_3C\theta_{123} \\ S\theta_{123} & C\theta_{123} & 0 & a_1S\theta_1 + a_2S\theta_{12} + a_3S\theta_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



Example: 3R Planar Arm [2]



Forward Kinematics

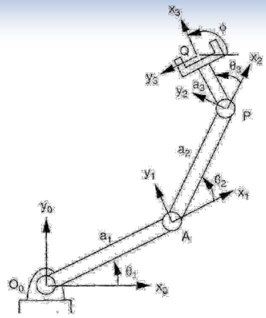
(solve for **x given θ** → **x** = f(**θ**))

Fairly straight forward:

$$ {}^{0}R_3 = \begin{bmatrix} C\theta_{123} & -S\theta_{123} & 0 \\ S\theta_{123} & C\theta_{123} & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}P_3 = \begin{bmatrix} a_1C\theta_1 + a_2C\theta_{12} + a_3C\theta_{123} \\ a_1S\theta_1 + a_2S\theta_{12} + a_3S\theta_{123} \\ 0 \end{bmatrix} $$

Example: 3R Planar Arm [3]



Inverse Kinematics
(solve for **θ given x** → $\mathbf{x} = f(\boldsymbol{\theta})$)

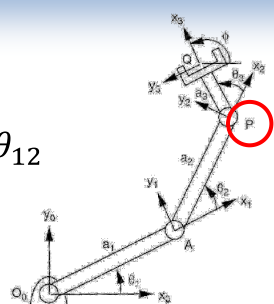
- Start with orientation φ:

$$C\theta_{123} = C\phi,\ S\theta_{123} = S\phi$$
$$\Rightarrow \theta_{123} = \theta_1 + \theta_2 + \theta_3 = \phi$$

- Get overall position $\boldsymbol{q} = [q_x \quad q_y]$:

$$q_x - a_3 C\phi = a_1 C\theta_1 + a_2 C\theta_{12}$$
$$q_y - a_3 S\phi = a_1 S\theta_1 + a_2 S\theta_{12} \ \ldots$$

Example: 3R Planar Arm [4]



- Introduce $\boldsymbol{p} = [p_x \quad p_y]$ before "wrist"

$$p_x = a_1 C\theta_1 + a_2 C\theta_{12}, p_y = a_1 S\theta_1 + a_2 S\theta_{12}$$
$$\Rightarrow p_x^2 + p_y^2 = a_1^2 + a_2^2 + 2a_1 a_2 C\theta_2$$

- Solve for θ_2:

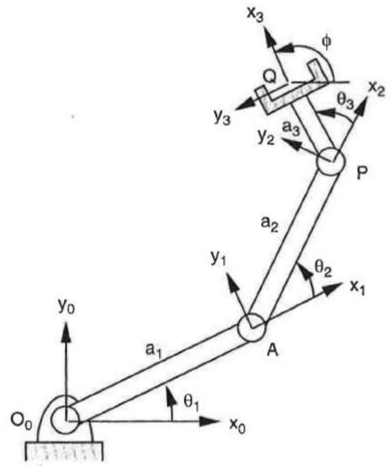
$$\theta_2 = \cos^{-1}\kappa,\ \kappa = \frac{p_x^2 + p_y^2 - a_1^2 - a_2^2}{2a_1 a_2}$$ (2 ℝ roots if |κ|<1)

- Solve for θ_1:

$$C\theta_1 = \frac{p_x(a_1 + a_2 C\theta_2) + p_y a_2 S\theta_2}{a_1^2 + a_2^2 + 2a_1 a_2 C\theta_2},\ S\theta_1 = \frac{-p_x a_2 S\theta_2 + p_y(a_1 + a_2 C\theta_2)}{a_1^2 + a_2^2 + 2a_1 a_2 C\theta_2}$$
$$\theta_1 = atan2(S\theta_1, C\theta_1)$$

Inverse Kinematics: Example I

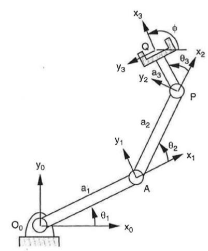
Planar Manipulator:



Inverse Kinematics: Example I

- Forward Kinematics:

[For the Frame {Q} at the end effector]:



$$\begin{bmatrix} q_x \\ q_y \\ q_z \\ 1 \end{bmatrix} = {}^0A_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 c\theta_1 + a_2 c\theta_{12} + a_3 c\theta_{123} \\ a_1 s\theta_1 + a_2 s\theta_{12} + a_3 s\theta_{123} \\ 0 \\ 1 \end{bmatrix}$$

$\because$

$$^0A_3 = \begin{bmatrix} c\theta_{123} & -s\theta_{123} & 0 & a_1 c\theta_1 + a_2 c\theta_{12} + a_3 c\theta_{123} \\ s\theta_{123} & c\theta_{123} & 0 & a_1 s\theta_1 + a_2 s\theta_{12} + a_3 s\theta_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

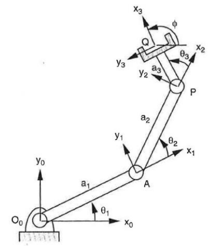
- For an arbitrary point **G** in the end effector: $^3\mathbf{g} = [g_u, g_v, 0, 1]^T$

$$\begin{bmatrix} g_x \\ g_y \\ g_z \\ 1 \end{bmatrix} = {}^0A_3 \begin{bmatrix} g_u \\ g_v \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} g_u c\theta_{123} - g_v s\theta_{123} + a_1 c\theta_1 + a_2 c\theta_{12} + a_3 c\theta_{123} \\ g_u s\theta_{123} + g_v c\theta_{123} + a_1 s\theta_1 + a_2 s\theta_{12} + a_3 s\theta_{123} \\ 0 \\ 1 \end{bmatrix}$$

Inverse Kinematics: Example I

- Forward Kinematics:

[For the Frame {Q} at the end effector]:



$$\begin{bmatrix} q_x \\ q_y \\ q_z \\ 1 \end{bmatrix} = {}^0A_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 c\theta_1 + a_2 c\theta_{12} + a_3 c\theta_{123} \\ a_1 s\theta_1 + a_2 s\theta_{12} + a_3 s\theta_{123} \\ 0 \\ 1 \end{bmatrix}$$

$\because$

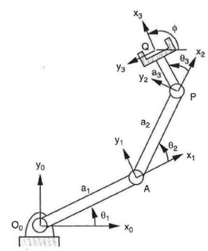
$${}^0A_3 = \begin{bmatrix} c\theta_{123} & -s\theta_{123} & 0 & a_1 c\theta_1 + a_2 c\theta_{12} + a_3 c\theta_{123} \\ s\theta_{123} & c\theta_{123} & 0 & a_1 s\theta_1 + a_2 s\theta_{12} + a_3 s\theta_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- For an arbitrary point **G** in the end effector: ${}^3\mathbf{g} = [g_u, g_v, 0, 1]^T$

$$\begin{bmatrix} g_x \\ g_y \\ g_z \\ 1 \end{bmatrix} = {}^0A_3 \begin{bmatrix} g_u \\ g_v \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} g_u c\theta_{123} - g_v s\theta_{123} + a_1 c\theta_1 + a_2 c\theta_{12} + a_3 c\theta_{123} \\ g_u s\theta_{123} + g_v c\theta_{123} + a_1 s\theta_1 + a_2 s\theta_{12} + a_3 s\theta_{123} \\ 0 \\ 1 \end{bmatrix}$$

Inverse Kinematics: Example I

- Inverse Kinematics:
  - Set the final position equal to the Forward Transformation Matrix ${}^0\mathbf{A}_3$:



$${}^0A_3 = \begin{bmatrix} c\phi & -s\phi & 0 & q_x \\ s\phi & c\phi & 0 & q_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

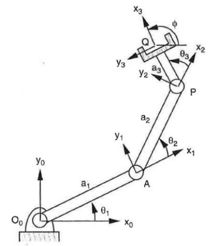
- The solution strategy is to equate the elements of ${}^0\mathbf{A}_3$ to that of the given position (q_x, q_y) and orientation ϕ

Inverse Kinematics: Example I

- Orientation (ϕ):

$$c\theta_{123} = c\phi,$$
$$s\theta_{123} = s\phi.$$

$$\theta_{123} = \theta_1 + \theta_2 + \theta_3 = \phi.$$



- Now Position of the 2DOF point **P**:

$$p_x = a_1 c\theta_1 + a_2 c\theta_{12},$$
$$p_y = a_1 s\theta_1 + a_2 s\theta_{12},$$

$\therefore \quad p_x = q_x - a_3 c\phi \qquad p_y = q_y - a_3 s\phi$

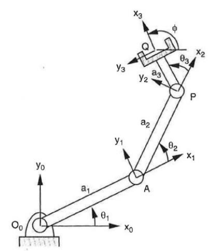
- Substitute: θ_3 disappears and now we can eliminate θ_1:

$$p_x^2 + p_y^2 = a_1^2 + a_2^2 + 2a_1 a_2 c\theta_2.$$

Inverse Kinematics: Example I

- we can eliminate θ_1…

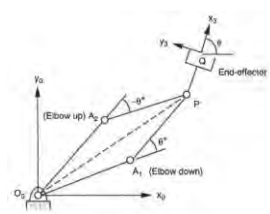
$$p_x^2 + p_y^2 = a_1^2 + a_2^2 + 2a_1 a_2 c\theta_2.$$



- Then solve for θ_{12}:

$$\theta_2 = \cos^{-1}\kappa, \qquad \kappa = \frac{p_x^2 + p_y^2 - a_1^2 - a_2^2}{2a_1 a_2}$$

  - This gives 2 real ($\mathbb{R}$) roots if $|\kappa| < 1$
  - One double root if $|\kappa| = 1$
  - No real roots if $|\kappa| > 1$
- Elbow up/down:
  - In general, **if** θ_2 is a solution **then** $-\theta_2$ is a solution



Inverse Kinematics: Example I

- Solving for θ_1…
  - Corresponding to each θ_2, we can solve θ_1

$$(a_1 + a_2 c\theta_2)c\theta_1 - (a_2 s\theta_2)s\theta_1 = p_x$$

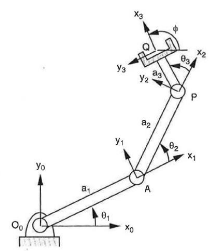
$$(a_2 s\theta_2)c\theta_1 + (a_1 + a_2 c\theta_2)s\theta_1 = p_y$$

$$c\theta_1 = \frac{p_x(a_1 + a_2 c\theta_2) + p_y a_2 s\theta_2}{\Delta},$$

$$s\theta_1 = \frac{-p_x a_2 s\theta_2 + p_y(a_1 + a_2 c\theta_2)}{\Delta}$$

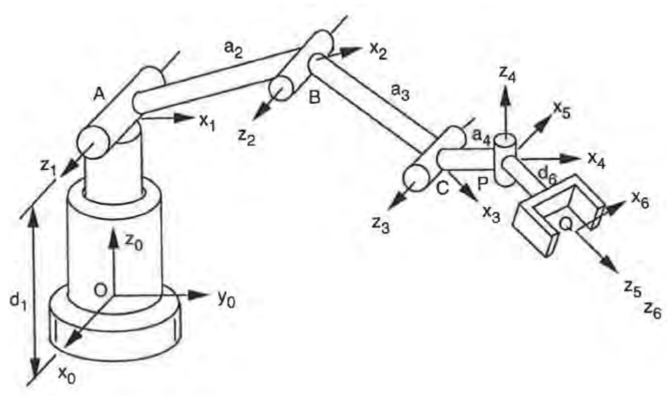
$$\Delta = a_1^2 + a_2^2 + 2a_1 a_2 c\theta_2$$

$$\boxed{\theta_1 = \mathrm{Atan2}(s\theta_1, c\theta_1)}$$



Inverse Kinematics: Example II

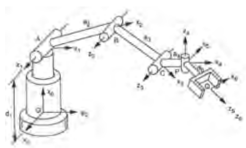
Elbow Manipulator:



Inverse Kinematics: Example II

- Target Position:

$\mathbf{u} = [u_x, u_y, u_z]^T$, $\mathbf{v} = [v_x, v_y, v_z]^T$, $\mathbf{w} = [w_x, w_y, w_z]^T$, and $\mathbf{p} = [p_x, p_y, p_z]^T$.



- Transformation Matrices:

$$A_1 = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad ({}^0A_1)^{-1} = \begin{bmatrix} c\theta_1 & s\theta_1 & 0 & 0 \\ -s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

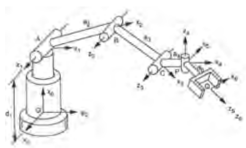
$$A_2 = \begin{bmatrix} c\theta_2 & 0 & -s\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ s\theta_2 & 0 & c\theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} c\theta_3 & 0 & -s\theta_3 & a_2(1 - c\theta_3) \\ 0 & 1 & 0 & 0 \\ s\theta_3 & 0 & c\theta_3 & -a_2 s\theta_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c\theta_4 & 0 & -s\theta_4 & (a_2 + a_3)(1 - c\theta_4) \\ 0 & 1 & 0 & 0 \\ s\theta_4 & 0 & c\theta_4 & -(a_2 + a_3)s\theta_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_5 = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & (a_2 + a_3 + a_4)(1 - c\theta_5) \\ s\theta_5 & c\theta_5 & 0 & -(a_2 + a_3 + a_4)s\theta_5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse Kinematics: Example II

- Key Matrix Products:

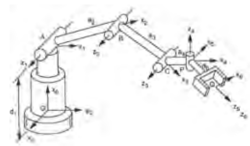


$$A_2A_3A_4 = \begin{bmatrix} c\theta_{234} & 0 & -s\theta_{234} & a_2c\theta_2 + a_3c\theta_{23} - (a_2 + a_3)c\theta_{234} \\ 0 & 1 & 0 & 0 \\ s\theta_{234} & 0 & c\theta_{234} & a_2s\theta_2 + a_3s\theta_{23} - (a_2 + a_3)s\theta_{234} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_1A_2A_3A_4 = \begin{bmatrix} c\theta_1c\theta_{234} & -s\theta_1 & -c\theta_1s\theta_{234} & c\theta_1[a_2c\theta_2 + a_3c\theta_{23} - (a_2 + a_3)c\theta_{234}] \\ s\theta_1c\theta_{234} & c\theta_1 & -s\theta_1s\theta_{234} & s\theta_1[a_2c\theta_2 + a_3c\theta_{23} - (a_2 + a_3)c\theta_{234}] \\ s\theta_{234} & 0 & c\theta_{234} & [a_2s\theta_2 + a_3s\theta_{23} - (a_2 + a_3)s\theta_{234}] \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse Kinematics: Example II

- Inverse Kinematics:



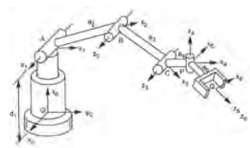
$$\mathbf{p} = A_1 A_2 A_3 A_4 \mathbf{p}_0.$$

$$A_1^{-1} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = A_2 A_3 A_4 \begin{bmatrix} a_2 + a_3 + a_4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$p_x c\theta_1 + p_y s\theta_1 = a_2 c\theta_2 + a_3 c\theta_{23} + a_4 c\theta_{234},$$
$$-p_x s\theta_1 + p_y c\theta_1 = 0,$$
$$p_z = a_2 s\theta_2 + a_3 s\theta_{23} + a_4 s\theta_{234}.$$

Inverse Kinematics: Example II

- Solving the System:



$$\theta_1 = \tan^{-1} \frac{p_y}{p_x}.$$

$$\theta_5 = \sin^{-1}(-w_x s\theta_1 + w_y c\theta_1).$$

$$\theta_{234} = \text{Atan2}\left[w_z / c\theta_5, (w_x c\theta_1 + w_y s\theta_1)/c\theta_5\right].$$

$$a_2 c\theta_2 + a_3 c\theta_{23} = k_1, \quad k_1 = p_x c\theta_1 + p_y s\theta_1 - a_4 c\theta_{234}$$
$$a_2 s\theta_2 + a_3 s\theta_{23} = k_2, \quad k_2 = p_z - a_4 s\theta_{234}$$

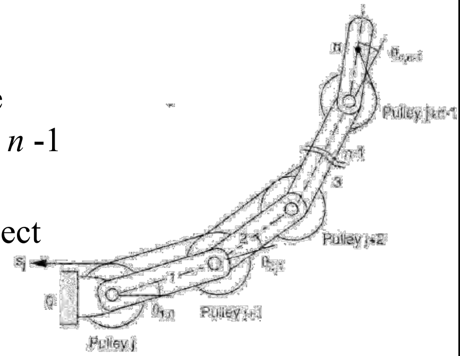
$$a_2^2 + a_3^2 + 2a_2 a_3 c\theta_3 = k_1^2 + k_2^2.$$

$$\theta_3 = \cos^{-1} \frac{k_1^2 + k_2^2 - a_2^2 - a_3^2}{2a_2 a_3}.$$

$$\theta_6 = \text{Atan2}(s\theta_6, c\theta_6).$$

Advanced Concept: Tendon-Driven Manipulators

- Tendons may be modelled as a transmission line
- in which the links are labeled sequentially from 0 to n and the pulleys are labeled from *j* to *j* + *n* -1
- Let θ_{ji} denote the angular displacement of link *j* with respect to link *i*.
- We can write a circuit equation once for each pulley pair as follows:



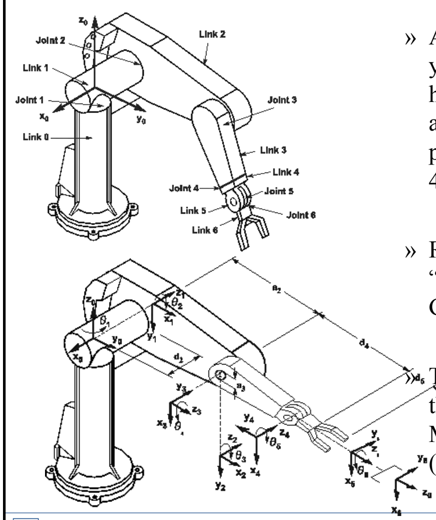
$$r_{j+i-1}\theta_{j+i-1,i} = \pm r_{j+i}\theta_{j+i,i} \qquad \text{for } i = 1, 2, \ldots, n-1.$$

$$\theta_{j+i-1,i} = \theta_{j+i-1,i-1} - \theta_{i,i-1} \qquad \text{for } i = 1, 2, \ldots, n.$$

$$\theta_{j,0} = \theta_{1,0} \pm (r_{j+1}/r_j)\theta_{2,1} \pm \cdots \pm (r_{j+n-1}/r_j)\theta_{n,n-1}.$$

Inverse Kinematics

- What about a more complicated mechanism?



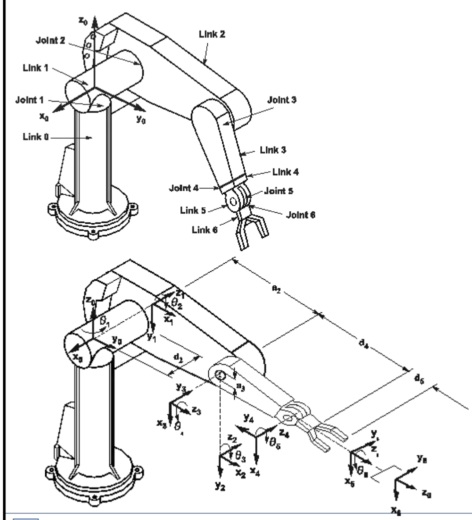
» A sufficient condition for a serial manipulator to yield a closed-form inverse kinematics solution is to have any three consecutive joint axes intersecting at a common point or any three consecutive joint axes parallel to each other. (Pieper and Roth (1969) via 4×4 matrix method)

» Raghavan and Roth 1990
"Kinematic Analysis of the 6R Manipulator of General Geometry"

» Tsai and Morgan 1985, "Solving the Kinematics of the Most General Six and Five-Dcgree-of-Freedom Manipulators by Continuation Methods" (posted online)

Inverse Kinematics

- What about a more complicated mechanism?



$$ {}^0\boldsymbol{T}_6 = {}^0\boldsymbol{T}_1\,{}^1\boldsymbol{T}_2\,{}^2\boldsymbol{T}_3\,{}^3\boldsymbol{T}_4\,{}^4\boldsymbol{T}_5\,{}^5\boldsymbol{T}_6 = \begin{pmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$
\begin{aligned}
n_x &= c_1(c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6) - s_1(s_4c_5c_6 + c_4s_6) \\
n_y &= s_1(c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6) + c_1(s_4c_5c_6 + c_4s_6) \\
n_z &= -s_{23}(c_4c_5c_6 - s_4s_6) - c_{23}s_5c_6 \\
s_x &= c_1(-c_{23}(c_4c_5s_6 + s_4c_6) + s_{23}s_5s_6) - s_1(-s_4c_5s_6 + c_4c_6) \\
s_y &= s_1(-c_{23}(c_4c_5s_6 + s_4c_6) + s_{23}s_5s_6) + c_1(-s_4c_5s_6 + c_4c_6) \\
s_z &= s_{23}(c_4c_5s_6 + s_4c_6) - c_{23}s_5s_6 \\
a_x &= c_1(c_{23}c_4s_5 + s_{23}c_5) - s_1s_4s_5 \\
a_y &= s_1(c_{23}c_4s_5 + s_{23}c_5) + c_1s_4s_5 \\
a_z &= -s_{23}c_4s_5 + c_{23}c_5) \\
p_x &= c_1(d_6(c_{23}c_4s_5 + s_{23}c_5) + s_{23}d_4 + a_3c_{23} + a_2c_2) - s_1(d_6s_4s_5 + d_2) \\
p_y &= s_1(d_6(c_{23}c_4s_5 + s_{23}c_5) + s_{23}d_4 + a_3c_{23} + a_2c_2) + c_1(d_6s_4s_5 + d_2) \\
p_z &= d_6(c_{23}c_5 - s_{23}c_4s_5) + c_{23}d_4 - a_3s_{23} - a_2s_2
\end{aligned}
$$

Symmetrical Parallel Manipulator

A sub-class of Parallel Manipulator:

- # Limbs (***m***) = # DOF (***F***)
- The joints are arranged in an identical pattern
- The # and location of actuated joints are the same

Thus:

- Number of Loops (L): One less than # of limbs

$$ L = m - 1 = F - 1 $$

- Connectivity (C_k)

$$ \sum_{k=1}^{m} C_k = (\lambda + 1)\,F - \lambda $$

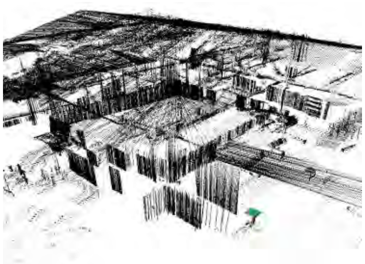
Where: λ: The DOF of the space that the system is in (e.g., λ=6 for 3D space).

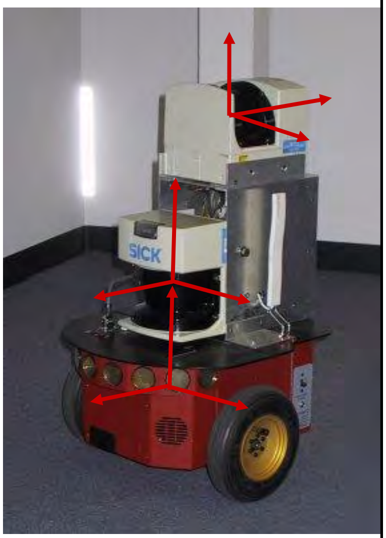
Mobile Platforms

- The preceding kinematic relationships are also important in mobile applications
- When we have sensors mounted on a platform, we need the ability to translate from the sensor frame into some world frame in which the vehicle is operating
- Should we just treat this as a P(*) mechanism?

Mobile Platforms [2]

- We typically assign a frame to the base of the vehicle
- Additional frames are assigned to the sensors
- We will develop these techniques in coming lectures





Summary

- Many ways to view a rotation
  - Rotation matrix
  - Euler angles
  - Quaternions
  - Direction Cosines
  - Screw Vectors

- Homogenous transformations
  - Based on homogeneous coordinates

Generalizing

Special Orthogonal & Special Euclidean Lie Algebras

- SO(n): R**o**tations

$$SO(n) = \{R \in \mathbb{R}^{n \times n} : RR^T = I, \det R = +1\}.$$

$$\exp(\widehat{\omega}\theta) = e^{\widehat{\omega}\theta} = I + \theta\widehat{\omega} + \frac{\theta^2}{2!}\widehat{\omega}^2 + \frac{\theta^3}{3!}\widehat{\omega}^3 + \dots$$

- SE(n): Transformations of EUCLIDEAN space

$$SE(n) := \mathbb{R}^n \times SO(n).$$

$$SE(3) = \{(p, R) : p \in \mathbb{R}^3, R \in SO(3)\} = \mathbb{R}^3 \times SO(3).$$

Projective Transformations …

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $\mathbf{l}_\infty$.
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, **I**, **J** (see section 2.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area

p.44, R. Hartley and A. Zisserman. *Multiple View Geometry in Computer Vision*

Homogenous Coordinates

$$\hat{p} = \begin{bmatrix} \rho p_x & \rho p_y & \rho p_z & \rho \end{bmatrix}^T$$

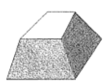
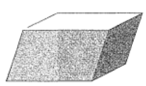
- ρ is a scaling value

Homogenous Transformation

$$\begin{bmatrix} {}^{A}R_{B} & {}^{A}p \\ \gamma & \rho \end{bmatrix}$$

- γ is a projective transformation
- The Homogenous Transformation is a **linear operation** (even if projection is not)

Projective Transformations & Other Transformations of 3D Space

Group	Matrix	Distortion	Invariant properties
Projective 15 dof	$\begin{bmatrix} A & \mathbf{t} \\ \mathbf{v}^{\mathsf{T}} & v \end{bmatrix}$		Intersection and tangency of surfaces in contact. Sign of Gaussian curvature.
Affine 12 dof	$\begin{bmatrix} A & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$		Parallelism of planes, volume ratios, centroids. The plane at infinity, π_∞, (see section 3.5).
Similarity 7 dof	$\begin{bmatrix} sR & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$		The absolute conic, Ω_∞, (see section 3.6).
Euclidean 6 dof	$\begin{bmatrix} R & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$		Volume.

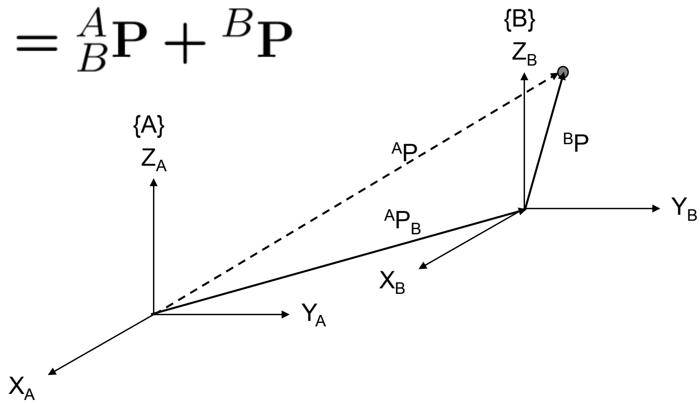
p.78, R. Hartley and A. Zisserman. *Multiple View Geometry in Computer Vision*

Coordinate Transformations [1]

- Translation Again:

  If {B} is translated with respect to {A} **without rotation**, then it is a vector sum

$$ {}^{A}\mathbf{P} = {}^{A}_{B}\mathbf{P} + {}^{B}\mathbf{P} $$



Coordinate Transformations [2]

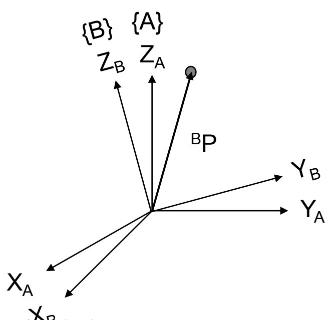
- Rotation Again:

  {B} is rotated with respect to {A} then
  use rotation matrix to determine new components

$$ {}^{A}\mathbf{P} = {}^{A}_{B}\mathbf{R}\,{}^{B}\mathbf{P} $$

- NOTE:
  - The Rotation matrix's ***subscript*** matches the position vector's **superscript**

$$ {}^{A}\mathbf{P} = {}^{A}_{[\![B]\!]}\mathbf{R}\,{}^{[\![B]\!]}\mathbf{P} $$

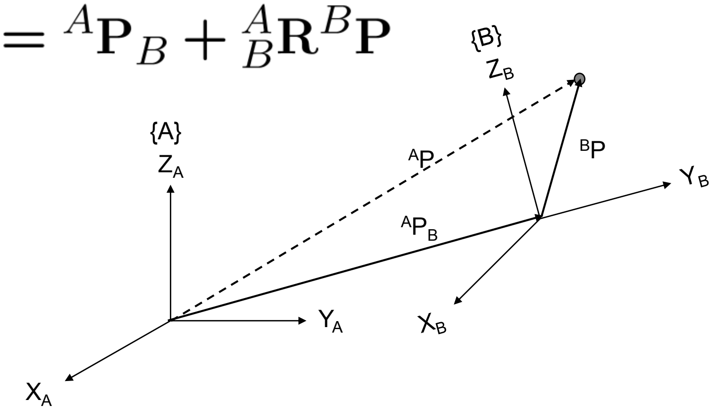


  - This gives Point Positions of {B} ORIENTED in {A}

Coordinate Transformations [3]

- Composite transformation:
  {B} is moved with respect to {A}:

$$ {}^{A}\mathbf{P} = {}^{A}\mathbf{P}_B + {}^{A}_{B}\mathbf{R}\,{}^{B}\mathbf{P} $$



General Coordinate Transformations [1]

- A compact representation of the translation and rotation is known as the **Homogeneous Transformation**

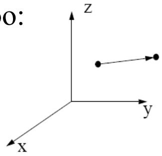
$$ {}^{A}_{B}\mathbf{T} = \begin{bmatrix} & {}^{A}_{B}\mathbf{R} & & {}^{A}\mathbf{P}_B \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

- This allows us to cast the rotation and translation of the general transform in a single matrix form

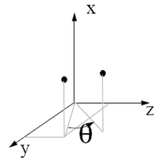
$$ \begin{bmatrix} {}^{A}\mathbf{P} \\ 1 \end{bmatrix} = {}^{A}_{B}\mathbf{T} \begin{bmatrix} {}^{B}\mathbf{P} \\ 1 \end{bmatrix} $$

General Coordinate Transformations [2]

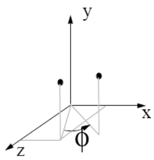
- Similarly, fundamental orthonormal transformations can be represented in this form too:



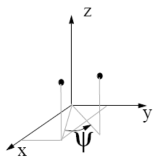
$$Trans(u, v, w) = \begin{bmatrix} 1 & 0 & 0 & u \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & w \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$Rotx(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\theta & -s\theta & 0 \\ 0 & s\theta & c\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$Roty(\phi) = \begin{bmatrix} c\phi & 0 & s\phi & 0 \\ 0 & 1 & 0 & 0 \\ -s\phi & 0 & c\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

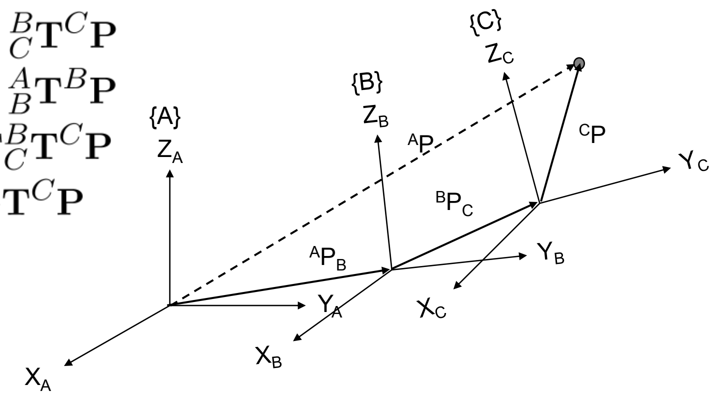


$$Rotz(\psi) = \begin{bmatrix} c\psi & -s\psi & 0 & 0 \\ s\psi & c\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

General Coordinate Transformations [3]

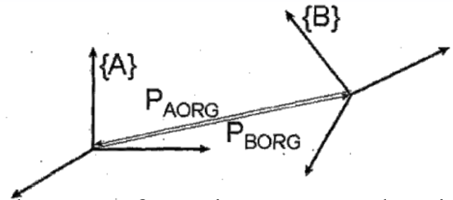
- Multiple transformations compounded as a chain

$$\begin{aligned} {}^{B}\mathbf{P} &= {}^{B}_{C}\mathbf{T}\,{}^{C}\mathbf{P} \\ {}^{A}\mathbf{P} &= {}^{A}_{B}\mathbf{T}\,{}^{B}\mathbf{P} \\ &= {}^{A}_{B}\mathbf{T}\,{}^{B}_{C}\mathbf{T}\,{}^{C}\mathbf{P} \\ &= {}^{A}_{C}\mathbf{T}\,{}^{C}\mathbf{P} \end{aligned}$$



$$ {}^{A}_{C}\mathbf{T} = \begin{bmatrix} {}^{A}_{B}\mathbf{R}\,{}^{B}_{C}\mathbf{R} & {}^{A}\mathbf{P}_{B} + {}^{A}_{B}\mathbf{R}\,{}^{B}\mathbf{P}_{C} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} $$

Inverse of a Homogeneous Transformation Matrix



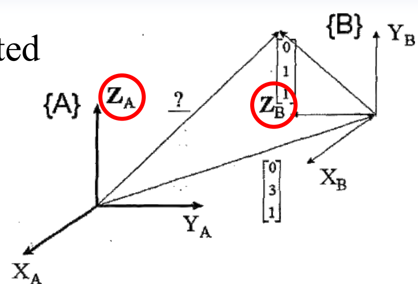
- The inverse of the transform is **not** equal to its transpose because this 4×4 matrix is not orthonormal ($T^{-1} \neq T^T$)
- Invert by parts to give:

$$ {}^A_B T = \begin{bmatrix} & {}^A_B R & & {}^A\mathbf{p}_{Borg/O_A} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^A_B T^{-1} = {}^B_A T = \begin{bmatrix} & {}^A_B R^T & & -{}^A_B R^T \cdot {}^A\mathbf{p}_{Borg/O_A} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & {}^B_A R & & {}^B\mathbf{p}_{Aorg/O_B} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Tutorial Problem

The origin of frame *{B}* is translated to a position [0 3 1] with respect to frame *{A}*.



We would like to find:

1. The homogeneous transformation between the two frames in the figure.
2. For a point *P* defined as as [0 1 1] in frame *{B}*, we would like to find the vector describing this point with respect to frame *{A}*.

Tutorial Solution

✔

• The matrix ${}_BT^A$ is formed as defined earlier:

• Since P in the frame is:

• We find vector **p** in frame *{A}* using the relationship

➔

- The matrix ${}_BT^A$ is formed as defined earlier:

$$ {}^A_BT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

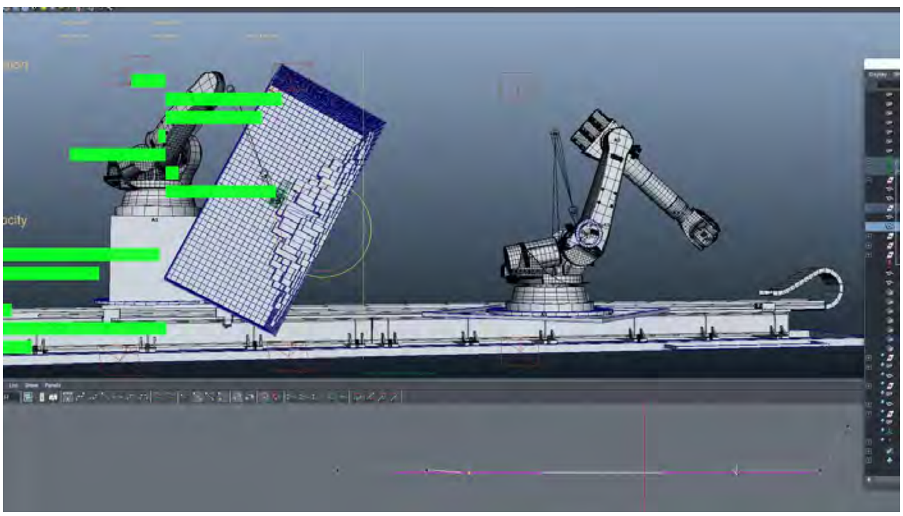
- Since P in the frame is: ${}^B\mathbf{p} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

- We find vector **p** in frame *{A}* using the relationship

$$ {}^Ap = {}^A_BT\, {}^Bp $$

➔ ${}^A\mathbf{p} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix}$

Cool Robotics Share



Robot

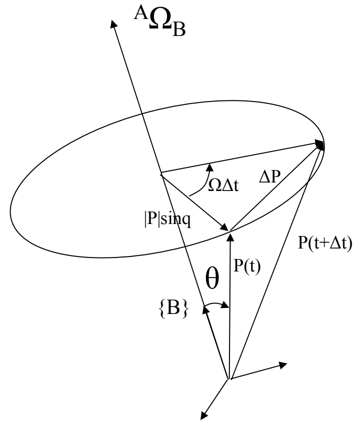
Dynamics

Robot Dynamics

Angular Velocity

- If we look at a small timeslice as a frame rotates with a moving point, we find

$$|\Delta \mathbf{P}| = \left(\left|{}^A\mathbf{P}\right| \sin\theta\right)\left(\left|{}^A\Omega_B\right|\Delta t\right)$$
$$\frac{|\Delta \mathbf{P}|}{\Delta t} = \left(\left|{}^A\mathbf{P}\right| \sin\theta\right)\left(\left|{}^A\Omega_B\right|\right)$$
$$= {}^A\Omega_B \times {}^A\mathbf{P}$$



$${}^A\mathbf{V}_P = {}^A\Omega_B \times {}^A R_B {}^B\mathbf{P}$$

Velocity

- Recall that we can specify a point in one frame relative to another as

$${}^A\mathbf{P} = {}^A\mathbf{P}_B + {}^A_B\mathbf{R}\,{}^B\mathbf{P}$$

- Differentiating w/r/t to **t** we find

$${}^A\mathbf{V}_P = \frac{d}{dt}{}^A\mathbf{P} = \lim_{\Delta t \to 0} \frac{{}^A\mathbf{P}(t+\Delta t) - {}^A\mathbf{P}(t)}{\Delta t}$$
$$= {}^A\dot{\mathbf{P}}_B + {}^A_B\mathbf{R}\,{}^B\dot{\mathbf{P}} + {}^A_B\dot{\mathbf{R}}\,{}^B\mathbf{P}$$

- This can be rewritten as

$${}^A\mathbf{V}_P = {}^A\mathbf{V}_{BORG} + {}^A\mathbf{R}_B {}^B\mathbf{V}_P + {}^A\Omega_B \times {}^A\mathbf{R}_B {}^B\mathbf{P}$$

Skew – Symmetric Matrix

$$\mathbf{V} = \omega \times \mathbf{r}$$

$$\Omega = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

$$\rightarrow \mathbf{V} = \Omega \mathbf{r}$$

Velocity Representations

- Euler Angles
  - For Z-Y-X (α,β,γ):

$$\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = \begin{pmatrix} -S\beta & 0 & 1 \\ C\beta S\gamma & C\gamma & 0 \\ C\beta C\gamma & -S\beta & 0 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

- Quaternions

$$\begin{pmatrix} \dot{\varepsilon}_0 \\ \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \\ \dot{\varepsilon}_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \varepsilon_1 & -\varepsilon_2 & -\varepsilon_3 \\ \varepsilon_0 & \varepsilon_3 & -\varepsilon_2 \\ -\varepsilon_3 & \varepsilon_0 & \varepsilon_1 \\ \varepsilon_2 & -\varepsilon_1 & \varepsilon_0 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

Manipulator Velocities

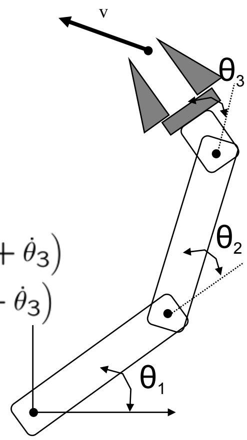
- Consider again the schematic of the planar manipulator shown. We found that the end effector position is given by

$$x = L_1 \cos\theta_1 + L_2 \cos(\theta_1+\theta_2) + L_3 \cos(\theta_1+\theta_2+\theta_3)$$
$$y = L_1 \sin\theta_1 + L_2 \sin(\theta_1+\theta_2) + L_3 \sin(\theta_1+\theta_2+\theta_3)$$

- Differentiating w/r/t to t

$$\dot{x} = -L_1\, s_1\, \dot\theta_1 - L_2\, s_{12}\left(\dot\theta_1+\dot\theta_2\right) - L_3\, s_{123}\left(\dot\theta_1+\dot\theta_2+\dot\theta_3\right)$$
$$\dot{y} = L_1\, c_1\, \dot\theta_1 + L_2\, c_{12}\left(\dot\theta_1+\dot\theta_2\right) + L_3\, c_{123}\left(\dot\theta_1+\dot\theta_2+\dot\theta_3\right)$$

- This gives the end effector velocity as a function of pose and joint velocities



Manipulator Velocities [2]

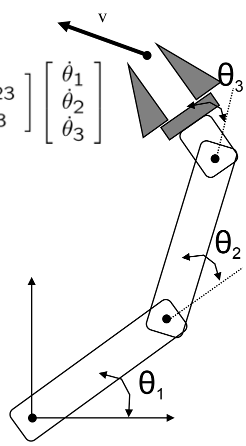
- Rearranging, we can recast this relation in matrix form

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix} -L_1 s_1 - L_2 s_{12} - L_3 s_{123} & -L_2 s_{12} - L_3 s_{123} & -L_3 s_{123} \\ L_1 c_1 + L_2 c_{12} + L_3 c_{123} & L_2 c_{12} + L_3 c_{123} & L_3 c_{123}\end{bmatrix} \begin{bmatrix}\dot\theta_1\\ \dot\theta_2\\ \dot\theta_3\end{bmatrix}$$

- Or

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial\theta_1} & \frac{\partial x}{\partial\theta_2} & \frac{\partial x}{\partial\theta_3} \\ \frac{\partial y}{\partial\theta_1} & \frac{\partial y}{\partial\theta_2} & \frac{\partial y}{\partial\theta_3}\end{bmatrix} \begin{bmatrix}\dot\theta_1\\ \dot\theta_2\\ \dot\theta_3\end{bmatrix}$$

- The resulting matrix is called the Jacobian and provides us with a mapping from Joint Space to Cartesian Space.



Moving On…Differential Motion

- Transformations also encode differential relationships
- Consider a manipulator (say 2DOF, RR)

$$x(\theta_1, \theta_2) = l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)$$
$$y(\theta_1, \theta_2) = l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)$$

- Differentiating with respect to the **angles** gives:

$$dx = \frac{\partial x(\theta_1, \theta_2)}{\partial \theta_1} d\theta_1 + \frac{\partial x(\theta_1, \theta_2)}{\partial \theta_2} d\theta_2$$

$$dy = \frac{\partial y(\theta_1, \theta_2)}{\partial \theta_1} d\theta_1 + \frac{\partial y(\theta_1, \theta_2)}{\partial \theta_2} d\theta_2$$

Differential Motion [2]

- Viewing this as a matrix → Jacobian

$$d\mathbf{x} = J d\theta$$

$$J = \begin{bmatrix} -l_1 \sin(\theta_1) - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

$$J = \begin{bmatrix} [J_1] & [J_2] \end{bmatrix}$$

$$v = J_1 \dot{\theta}_1 + J_2 \dot{\theta}_2$$

Infinitesimal Rotations

- $\cos(d\phi) = 1, \sin(d\phi) = d\phi$

$$\mathbf{R}_x(d\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cd\phi & -sd\phi \\ 0 & sd\phi & cd\phi \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -d\phi_x \\ 0 & d\phi_x & 1 \end{bmatrix}$$

$$\mathbf{R}_y(d\phi) = \begin{bmatrix} cd\phi & 0 & sd\phi \\ 0 & 1 & 0 \\ -sd\phi & 0 & cd\phi \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & d\phi_y \\ 0 & 1 & 0 \\ -d\phi_y & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}_z(d\phi) = \begin{bmatrix} cd\phi & -sd\phi & 0 \\ sd\phi & cd\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & -d\phi_z & 0 \\ d\phi_z & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Note that:

$$R_x(d\varphi)R_y(d\varphi) = R_y(d\varphi)R_x(d\varphi)$$

→ Therefore … they **commute**

Summary

- Many ways to handle motion
  - Direct Kinematics
  - Dynamics
- Homogenous transformations
  - Based on homogeneous coordinates