RAGHAVAN AND ROTH'S SOLUTION

The Sylvester dialytic elimination method described in Appendix B is effective for a small system of polynomial equations of relatively low degree. For more complicated polynomial systems, the method can make a problem unmanageably large. Therefore, it is desirable to develop alternative methods of generating new linearly independent equations that introduce no new power products, or at worst a small number of new power products. In this appendix we describe a technique employed by Raghavan and Roth (1990a,b) for solving the inverse kinematics of the general $6R$ manipulator.

C.1 LOOP-CLOSURE EQUATION

Let us consider the general $6R$ manipulator shown in Fig. 2.11. For convenience, we decompose ${}^1A_2$ as a product of two matrices:

$$ {}^1A_2 = {}^1G_2 \, {}^1H_2, \tag{C.1} $$

where

$$ {}^1G_2 = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & 0 \\ s\theta_2 & c\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad {}^1H_2 = \begin{bmatrix} 1 & 0 & 0 & a_2 \\ 0 & c\alpha_2 & -s\alpha_2 & 0 \\ 0 & s\alpha_2 & c\alpha_2 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$

Note that ${}^1G_2$ contains only the joint variable and ${}^1H_2$ contains only the link parameters.

Substituting Eq. (C.1) into (2.99), we obtain

$$ {}^0A_1\,{}^1G_2\,{}^1H_2\,{}^2A_3\,{}^3A_4\,{}^4A_5\,{}^5A_6 = {}^0A_6. \tag{C.2}$$

Premultipling both sides of Eq. (C.2) by $({}^0A_1\,{}^1G_2)^{-1}$ and postmultiplying both sides by ${}^5A_6^{-1}$, we obtain

$$ {}^1H_2\,{}^2A_3\,{}^3A_4\,{}^4A_5 = {}^1G_2^{-1}\,{}^0A_1^{-1}\,{}^0A_6\,{}^5A_6^{-1}. \tag{C.3}$$

Note that by moving θ_1, θ_2, and θ_6 to the right-hand side of the equation, we have effectively lowered the degrees of the equations. When the matrix multiplication is carried out, Eq. (C.3) takes the form

$$\begin{bmatrix} f_{11}(\theta_3,\theta_4,\theta_5) & f_{12}(\theta_3,\theta_4,\theta_5) & f_{13}(\theta_3,\theta_4,\theta_5) & f_{14}(\theta_3,\theta_4,\theta_5) \\ f_{21}(\theta_3,\theta_4,\theta_5) & f_{22}(\theta_3,\theta_4,\theta_5) & f_{23}(\theta_3,\theta_4,\theta_5) & f_{24}(\theta_3,\theta_4,\theta_5) \\ f_{31}(\theta_4,\theta_5) & f_{32}(\theta_4,\theta_5) & f_{33}(\theta_4,\theta_5) & f_{34}(\theta_4,\theta_5) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} f'_{11}(\theta_1,\theta_2,\theta_6) & f'_{12}(\theta_1,\theta_2,\theta_6) & f'_{13}(\theta_1,\theta_2) & f'_{14}(\theta_1,\theta_2) \\ f'_{21}(\theta_1,\theta_2,\theta_6) & f'_{22}(\theta_1,\theta_2,\theta_6) & f'_{23}(\theta_1,\theta_2) & f'_{24}(\theta_1,\theta_2) \\ f'_{31}(\theta_1,\theta_2,\theta_6) & f'_{32}(\theta_1,\theta_2,\theta_6) & f'_{33}(\theta_1,\theta_2) & f'_{14}(\theta_1,\theta_2) \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{C.4}$$

Equation (C.4) only reveals the variables appearing in the elements of Eq. (C.3). An examination of Eq. (C.4) reveals that the six scalar equations obtained from the third and fourth columns are free of the variable θ_6. These six equations can be written in vector form, denoted as **a** and **b**, as follows:

$$\mathbf{a}: \begin{bmatrix} c\theta_3 & s\theta_3 & 0 \\ -c\alpha_2 s\theta_3 & c\alpha_2 c\theta_3 & s\alpha_2 \\ s\alpha_2 s\theta_3 & -s\alpha_2 c\theta_3 & c\alpha_2 \end{bmatrix} \begin{bmatrix} \mu_x \\ \mu_y \\ \mu_z \end{bmatrix} + \begin{bmatrix} a_2 \\ 0 \\ d_2 \end{bmatrix} = \begin{bmatrix} c\theta_2 & s\theta_2 & 0 \\ s\theta_2 & -c\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix}, \tag{C.5}$$

$$\mathbf{b}: \begin{bmatrix} c\theta_3 & s\theta_3 & 0 \\ -c\alpha_2 s\theta_3 & c\alpha_2 c\theta_3 & s\alpha_2 \\ s\alpha_2 s\theta_3 & -s\alpha_2 c\theta_3 & c\alpha_2 \end{bmatrix} \begin{bmatrix} \nu_x \\ \nu_y \\ \nu_z \end{bmatrix} = \begin{bmatrix} c\theta_2 & s\theta_2 & 0 \\ s\theta_2 & -c\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}, \tag{C.6}$$

where

$$\mu_x = g_x c\theta_4 + g_y s\theta_4 + a_3,$$
$$\mu_y = -(g_x s\theta_4 - g_y c\theta_4)c\alpha_3 + g_z s\alpha_3,$$
$$\mu_z = (g_x s\theta_4 - g_y c\theta_4)s\alpha_3 + g_z c\alpha_3 + d_3,$$

$$\nu_x = m_x c\theta_4 + m_y s\theta_4,$$
$$\nu_y = -(m_x s\theta_4 - m_y c\theta_4)c\alpha_3 + m_z s\alpha_3,$$
$$\nu_z = (m_x s\theta_4 - m_y c\theta_4)s\alpha_3 + m_z c\alpha_3,$$

and where g_x, g_y, g_z, h_x, h_y, h_z, m_x, m_y, m_z, n_x, n_y, and n_z are defined in Chapter 2 under Eqs. (2.106) and (2.107). We note that μ_x, μ_y, μ_z, ν_x, ν_y, and ν_z are linear functions of the terms $s\theta_4 s\theta_5$, $s\theta_4 c\theta_5$, $c\theta_4 s\theta_5$, $c\theta_4 c\theta_5$, $s\theta_4$, $c\theta_4$, $s\theta_5$, $c\theta_5$, and 1, whereas h_x, h_y, h_z, n_x, n_y, and n_z are linear functions of the terms $s\theta_1$, $c\theta_1$, and 1.

The two vectors **a** and **b** in Eqs. (C.5) and (C.6) represent six scalar equations in five unknowns: $\theta_1, \theta_2, \ldots, \theta_5$. To eliminate several variables at a time, we treat some of the power products as new variables with the other power products suppressed. Toward this end, we write Eqs. (C.5) and (C.6) in the matrix form

$$A \begin{bmatrix} s\theta_4 s\theta_5 \\ s\theta_4 c\theta_5 \\ c\theta_4 s\theta_5 \\ c\theta_4 c\theta_5 \\ s\theta_4 \\ c\theta_4 \\ s\theta_5 \\ c\theta_5 \\ 1 \end{bmatrix} = B \begin{bmatrix} s\theta_1 s\theta_2 \\ s\theta_1 c\theta_2 \\ c\theta_1 s\theta_2 \\ c\theta_1 c\theta_2 \\ s\theta_1 \\ c\theta_1 \\ s\theta_2 \\ c\theta_2 \end{bmatrix}, \tag{C.7}$$

where A is a 6×9 matrix whose elements are linear combinations of $s\theta_3$, $c\theta_3$, and 1, and B is a 6×8 matrix whose elements are all constants.

It can be shown that the products $\mathbf{a}^T\mathbf{a}$, $\mathbf{a}^T\mathbf{b}$, $\mathbf{a} \times \mathbf{b}$, and $(\mathbf{a}^T\mathbf{a})\mathbf{b} - 2(\mathbf{a}^T\mathbf{b})\mathbf{a}$ result in eight additional polynomials which take the same form as Eq. (C.7) (Raghavan and Roth, 1990a). Combining these eight equations with Eq. (C.7), we obtain 14 linearly independent equations, which can be written

$$A' \begin{bmatrix} s\theta_4 s\theta_5 \\ s\theta_4 c\theta_5 \\ c\theta_4 s\theta_5 \\ c\theta_4 c\theta_5 \\ s\theta_4 \\ c\theta_4 \\ s\theta_5 \\ c\theta_5 \\ 1 \end{bmatrix} = B' \begin{bmatrix} s\theta_1 s\theta_2 \\ s\theta_1 c\theta_2 \\ c\theta_1 s\theta_2 \\ c\theta_1 c\theta_2 \\ s\theta_1 \\ c\theta_1 \\ s\theta_2 \\ c\theta_2 \end{bmatrix}, \tag{C.8}$$

where A' is a 14×9 matrix whose elements are linear combinations of $s\theta_3$, $c\theta_3$, and 1, and B' is a 14×8 matrix whose elements are constants.

C.2 ELIMINATION OF θ_1 AND θ_2

In this section we show how θ_1 and θ_2 can be eliminated simultaneously from Eq. (C.8). Toward this goal, we treat $s\theta_1 s\theta_2$, $s\theta_1 c\theta_2$, $c\theta_1 s\theta_2$, $c\theta_1 c\theta_2$, $s\theta_1$, $c\theta_1$, $s\theta_2$, and $c\theta_2$ in Eq. (C.8) as eight independent variables, and the left-hand-side terms as constants. Then Eq. (C.8) represents 14 linearly independent equations in eight unknowns. We can solve these eight variables from any eight of the 14 equations and substitute them back into the remaining six equations. This results in six independent equations, free of θ_1 and θ_2, which can be arranged in matrix form:

$$E \begin{bmatrix} s\theta_4 s\theta_5 \\ s\theta_4 c\theta_5 \\ c\theta_4 s\theta_5 \\ c\theta_4 c\theta_5 \\ s\theta_4 \\ c\theta_4 \\ s\theta_5 \\ c\theta_5 \\ 1 \end{bmatrix} = [0], \tag{C.9}$$

where E is a 6×9 matrix whose elements are linear combinations of $s\theta_3$, $c\theta_3$, and 1.

C.3 ELIMINATION OF θ_4 AND θ_5

In this section we eliminate θ_4 and θ_5 simultaneously. We note that the six equations in Eq. (C.9) are already written with the variable θ_3 suppressed. We make use of the following trigonometric identities to convert the equations into polynomials.

$$c\theta_i = \frac{1 - t_i^2}{1 + t_i^2}, \tag{C.10}$$

$$s\theta_i = \frac{2t_i}{1 + t_i^2}, \tag{C.11}$$

where $t_i = \tan(\theta_i/2)$.

Substituting Eqs. (C.10) and (C.11) for $i = 4$ and 5 into Eq. (C.9) and then multiplying each equation by $(1 + t_4^2)(1 + t_5^2)$ to clear the denominators, we obtain

$$E' \begin{bmatrix} t_4^2 t_5^2 \\ t_4^2 t_5 \\ t_4^2 \\ t_4 t_5^2 \\ t_4 t_5 \\ t_4 \\ t_5^2 \\ t_5 \\ 1 \end{bmatrix} = [0], \tag{C.12}$$

where E' is a 6×9 matrix whose elements are linear combinations of $s\theta_3$, $c\theta_3$, and 1. Substituting Eqs. (C.10) and (C.11) for $i = 3$ into Eq. (C.12) and multiplying the first four resulting equations by $(1 + t_3^2)$, we obtain

$$E'' \begin{bmatrix} t_4^2 t_5^2 \\ t_4^2 t_5 \\ t_4^2 \\ t_4 t_5^2 \\ t_4 t_5 \\ t_4 \\ t_5^2 \\ t_5 \\ 1 \end{bmatrix} = [0], \tag{C.13}$$

where E'' is a 6×9 matrix. Note that the elements in the first four rows of E'' are quadratic in t_3, whereas the elements in the last two rows are rational functions of t_3, the numerators being quadratic polynomials in t_3 and the denominators being $(1 + t_3^2)$. Multiplying Eq. (C.13) by t_4 yields the following six additional linearly independent equations:

$$E'' \begin{bmatrix} t_4^3 t_5^2 \\ t_4^3 t_5 \\ t_4^3 \\ t_4^2 t_5^2 \\ t_4^2 t_5 \\ t_4^2 \\ t_4 t_5^2 \\ t_4 t_5 \\ t_4 \end{bmatrix} = [0]. \tag{C.14}$$

Finally, we combine Eqs. (C.13) and (C.14) in matrix form:

$$\begin{bmatrix} E'' & 0 \\ 0 & E'' \end{bmatrix} \begin{bmatrix} t_4^3 t_5^2 \\ t_4^3 t_5 \\ t_4^3 \\ t_4^2 t_5^2 \\ t_4^2 t_5 \\ t_4^2 \\ t_4 t_5^2 \\ t_4 t_5 \\ t_4 \\ t_5^2 \\ t_5 \\ 1 \end{bmatrix} = [0]. \tag{C.15}$$

We may consider $t_4^3 t_5^2$, $t_4^3 t_5$, t_4^3, $t_4^2 t_5^2$, $t_4^2 t_5$, t_4^2, $t_4 t_5^2$, $t_4 t_5$, t_4, t_5^2, t_5, and 1 as 12 unknowns. Then Eq. (C.15) constitutes a set of 12 linearly independent equations. The compatibility condition for nontrivial solution to exist is that the coefficient matrix must be singular. Setting the determinant of the coefficient matrix to zero yields a 16th-degree polynomial in t_3. See Raghavan and Roth (1990a) for a more detailed derivation of the equation. Once θ_3 is solved, the other variables can be solved by back substitution. We conclude that the inverse kinematics of the general $6R$ robot has at most 16 real solutions.

REFERENCES

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