Position and Orientation [3]

- The rotations can be analysed based on the unit components …
- That is: the components of the orientation matrix are the unit vectors projected **onto** the unit directions of the reference frame

$$ {}^{A}_{B}\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

$$ \begin{array}{c|ccc} {}^{A}_{B}R & (b_x)\,\hat{\imath}_B & (b_y)\,\hat{\jmath}_B & (b_z)\,\hat{k}_B \\ \hline (a_x)\,\hat{\imath}_A & \hat{\imath}_B \cdot \hat{\imath}_A & \hat{\jmath}_B \cdot \hat{\imath}_A & \hat{k}_B \cdot \hat{\imath}_A \\ (a_y)\,\hat{\jmath}_A & \hat{\imath}_B \cdot \hat{\jmath}_A & \hat{\jmath}_B \cdot \hat{\jmath}_A & \hat{k}_B \cdot \hat{\jmath}_A \\ (a_z)\,\hat{k}_A & \hat{\imath}_B \cdot \hat{k}_A & \hat{\jmath}_B \cdot \hat{k}_A & \hat{k}_B \cdot \hat{k}_A \end{array} $$

Homogenous Transformation

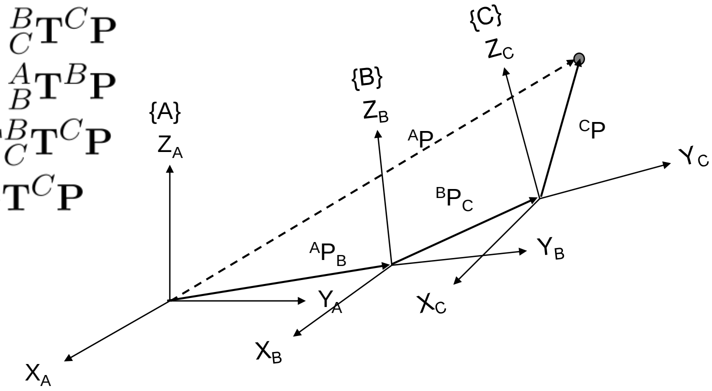
$$ \begin{bmatrix} {}^{A}R_{B} & {}^{A}p \\ \gamma & \rho \end{bmatrix} $$

- γ is a projective transformation
- The Homogenous Transformation is a **linear operation** (even if projection is not)

General Coordinate Transformations [3]

- Multiple transformations compounded as a chain

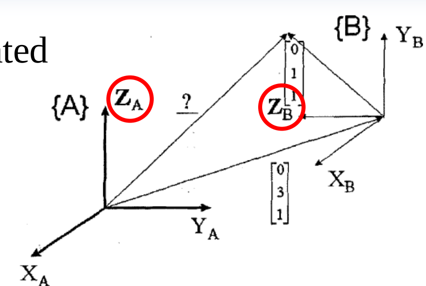
$$
\begin{aligned}
{}^{B}\mathbf{P} &= {}^{B}_{C}\mathbf{T}\,{}^{C}\mathbf{P} \\
{}^{A}\mathbf{P} &= {}^{A}_{B}\mathbf{T}\,{}^{B}\mathbf{P} \\
&= {}^{A}_{B}\mathbf{T}\,{}^{B}_{C}\mathbf{T}\,{}^{C}\mathbf{P} \\
&= {}^{A}_{C}\mathbf{T}\,{}^{C}\mathbf{P}
\end{aligned}
$$



$$
{}^{A}_{C}\mathbf{T} = \left[\begin{array}{cc} {}^{A}_{B}\mathbf{R}\,{}^{B}_{C}\mathbf{R} & {}^{A}\mathbf{P}_{B} + {}^{A}_{B}\mathbf{R}\,{}^{B}\mathbf{P}_{C} \\ 0 \quad 0 \quad 0 & 1 \end{array} \right]
$$

Tutorial Problem

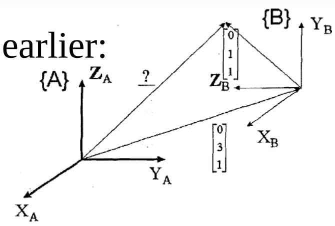
The origin of frame $\{B\}$ is translated to a position [0 3 1] with respect to frame $\{A\}$.



We would like to find:

1. The homogeneous transformation between the two frames in the figure.
2. For a point P defined as as [0 1 1] in frame $\{B\}$, we would like to find the vector describing this point with respect to frame $\{A\}$.

Tutorial Solution



- The matrix $_BT^A$ is formed as defined earlier:

$$^A_BT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

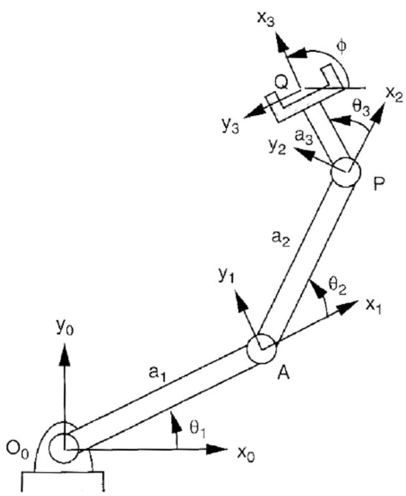
- Since P in the frame is: $^B\mathbf{p} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

- We find vector **p** in frame *{A}* using the relationship

$$^Ap = {}^A_BT\,{}^Bp$$

➔ $^A\mathbf{p} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix}$

Example: FK/IK of a 3R Planar Arm



- Derived from Tsai (p. 63)

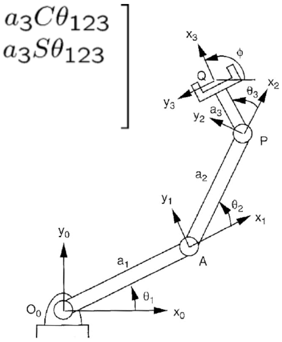
Example: 3R Planar Arm [2]

Position Analysis: 3·Planar 1-R Arm rotating about **Z** **[Ⓩ]**

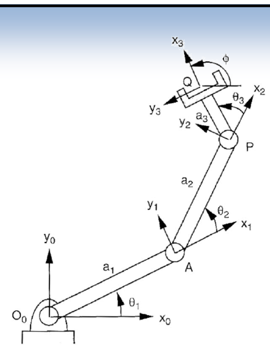
$${}^{0}A_3 = {}^{0}A_1 \cdot {}^{1}A_2 \cdot {}^{2}A_3$$

Substituting gives:

$${}^{0}A_3 = \begin{bmatrix} C\theta_{123} & -S\theta_{123} & 0 & a_1 C\theta_1 + a_2 C\theta_{12} + a_3 C\theta_{123} \\ S\theta_{123} & C\theta_{123} & 0 & a_1 S\theta_1 + a_2 S\theta_{12} + a_3 S\theta_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Example: 3R Planar Arm [2]



Forward Kinematics

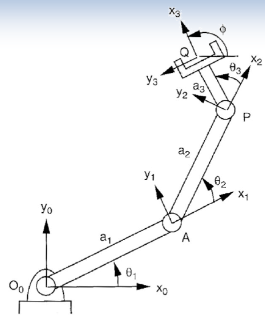
(solve for **x given θ** → **x** = *f* (**θ**))

Fairly straight forward:

$${}^{0}R_3 = \begin{bmatrix} C\theta_{123} & -S\theta_{123} & 0 \\ S\theta_{123} & C\theta_{123} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^{0}P_3 = \begin{bmatrix} a_1 C\theta_1 + a_2 C\theta_{12} + a_3 C\theta_{123} \\ a_1 S\theta_1 + a_2 S\theta_{12} + a_3 S\theta_{123} \\ 0 \end{bmatrix}$$

Example: 3R Planar Arm [3]



Inverse Kinematics
(solve for **θ given x** → **x** = *f* (**θ**))

- Start with orientation φ:

$$C\theta_{123} = C\phi,\ S\theta_{123} = S\phi$$

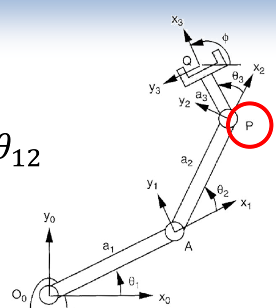
$$\Rightarrow \theta_{123} = \theta_1 + \theta_2 + \theta_3 = \phi$$

- Get overall position $\boldsymbol{q} = [q_x \quad q_y]$:

$$q_x - a_3 C\phi = a_1 C\theta_1 + a_2 C\theta_{12}$$

$$q_y - a_3 S\phi = a_1 S\theta_1 + a_2 S\theta_{12} \ \ldots$$

Example: 3R Planar Arm [4]



- Introduce $\boldsymbol{p} = [p_x \quad p_y]$ before "wrist"

$$p_x = a_1 C\theta_1 + a_2 C\theta_{12}, p_y = a_1 S\theta_1 + a_2 S\theta_{12}$$

$$\Rightarrow p_x^2 + p_y^2 = a_1^2 + a_2^2 + 2a_1 a_2 C\theta_2$$

- Solve for θ_2:

$$\theta_2 = \cos^{-1}\kappa,\ \kappa = \frac{p_x^2+p_y^2-a_1^2-a_2^2}{2a_1a_2}$$ (2 ℝ roots if |κ|<1)

- Solve for θ_1:

$$C\theta_1 = \frac{p_x(a_1+a_2C\theta_2)+p_ya_2S\theta_2}{a_1^2+a_2^2+2a_1a_2C\theta_2},\ S\theta_1 = \frac{-p_xa_2S\theta_2+p_y(a_1+a_2C\theta_2)}{a_1^2+a_2^2+2a_1a_2C\theta_2}$$

$$\theta_1 = atan2(S\theta_1, C\theta_1)$$