Two Inverse Kinematics Examples

METR 4202: Advanced Control & **Robotics**

Dr Surya Singh – Lecture 3 Supplementary

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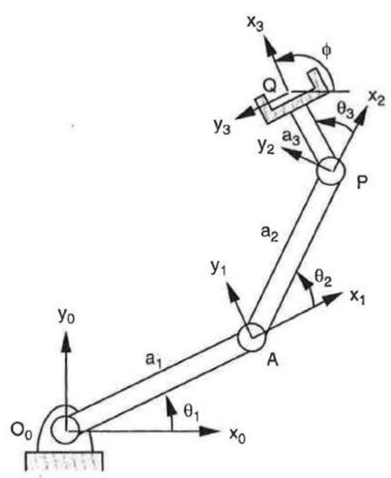
metr4202@itee.uq.edu.au

http://robotics.itee.uq.edu.au/~metr4202/

Inverse Kinematics: Example I

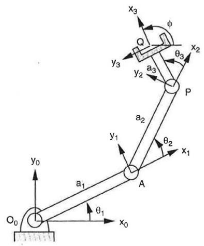
Planar Manipulator:



Inverse Kinematics: Example I

- Forward Kinematics:

[For the Frame {Q} at the end effector]:



$$\begin{bmatrix} q_x \\ q_y \\ q_z \\ 1 \end{bmatrix} = {}^0A_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 c\theta_1 + a_2 c\theta_{12} + a_3 c\theta_{123} \\ a_1 s\theta_1 + a_2 s\theta_{12} + a_3 s\theta_{123} \\ 0 \\ 1 \end{bmatrix}$$

$\because$

$${}^0A_3 = \begin{bmatrix} c\theta_{123} & -s\theta_{123} & 0 & a_1 c\theta_1 + a_2 c\theta_{12} + a_3 c\theta_{123} \\ s\theta_{123} & c\theta_{123} & 0 & a_1 s\theta_1 + a_2 s\theta_{12} + a_3 s\theta_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

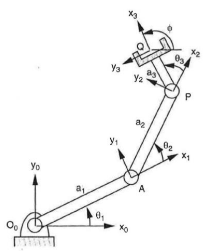
- For an arbitrary point **G** in the end effector: ${}^3\mathbf{g} = [g_u, g_v, 0, 1]^T$

$$\begin{bmatrix} g_x \\ g_y \\ g_z \\ 1 \end{bmatrix} = {}^0A_3 \begin{bmatrix} g_u \\ g_v \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} g_u c\theta_{123} - g_v s\theta_{123} + a_1 c\theta_1 + a_2 c\theta_{12} + a_3 c\theta_{123} \\ g_u s\theta_{123} + g_v c\theta_{123} + a_1 s\theta_1 + a_2 s\theta_{12} + a_3 s\theta_{123} \\ 0 \\ 1 \end{bmatrix}$$

Inverse Kinematics: Example I

- Forward Kinematics:

[For the Frame {Q} at the end effector]:



$$\begin{bmatrix} q_x \\ q_y \\ q_z \\ 1 \end{bmatrix} = {}^0A_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 c\theta_1 + a_2 c\theta_{12} + a_3 c\theta_{123} \\ a_1 s\theta_1 + a_2 s\theta_{12} + a_3 s\theta_{123} \\ 0 \\ 1 \end{bmatrix}$$

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- For an arbitrary point **G** in the end effector: ${}^3\mathbf{g} = [g_u, g_v, 0, 1]^T$

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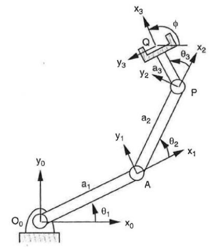
Inverse Kinematics: Example I

- Inverse Kinematics:
  - Set the final position equal to the Forward Transformation Matrix $^{\mathbf{0}}\mathbf{A_3}$:

$$^{0}A_3 = \begin{bmatrix} c\phi & -s\phi & 0 & q_x \\ s\phi & c\phi & 0 & q_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

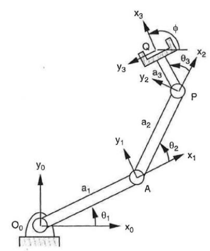


- The solution strategy is to equate the elements of $^{\mathbf{0}}\mathbf{A_3}$ to that of the given position (q_x, q_y) and orientation ϕ

Inverse Kinematics: Example I

- Orientation (ϕ):

$$c\theta_{123} = c\phi,$$
$$s\theta_{123} = s\phi.$$
$$\theta_{123} = \theta_1 + \theta_2 + \theta_3 = \phi.$$



- Now Position of the 2DOF point **P**:

$$p_x = a_1 c\theta_1 + a_2 c\theta_{12},$$
$$p_y = a_1 s\theta_1 + a_2 s\theta_{12},$$

$\therefore \quad p_x = q_x - a_3 c\phi \qquad p_y = q_y - a_3 s\phi$

- Substitute: θ_3 disappears and now we can eliminate θ_1:

$$p_x^2 + p_y^2 = a_1^2 + a_2^2 + 2a_1a_2c\theta_2.$$

Inverse Kinematics: Example I

- we can eliminate θ_1…

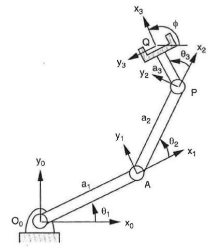
  $p_x^2 + p_y^2 = a_1^2 + a_2^2 + 2a_1a_2c\theta_2.$

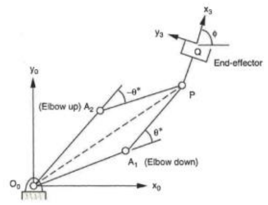
- Then solve for θ_{12}:

  $$\theta_2 = \cos^{-1}\kappa, \qquad \kappa = \frac{p_x^2 + p_y^2 - a_1^2 - a_2^2}{2a_1a_2}$$

  – This gives 2 real ($\mathbb{R}$) roots if $|\kappa| < 1$
  – One double root if $|\kappa| = 1$
  – No real roots if $|\kappa| > 1$

- Elbow up/down:
  – In general, **if** θ_2 is a solution **then** $-\theta_2$ is a solution





Inverse Kinematics: Example I

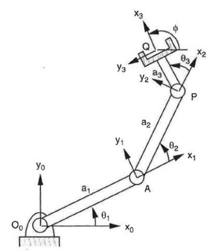
- Solving for θ_1…
  – Corresponding to each θ_2, we can solve θ_1

$$(a_1 + a_2c\theta_2)c\theta_1 - (a_2s\theta_2)s\theta_1 = p_x$$
$$(a_2s\theta_2)c\theta_1 + (a_1 + a_2c\theta_2)s\theta_1 = p_y$$

$$c\theta_1 = \frac{p_x(a_1 + a_2c\theta_2) + p_ya_2s\theta_2}{\Delta},$$
$$s\theta_1 = \frac{-p_xa_2s\theta_2 + p_y(a_1 + a_2c\theta_2)}{\Delta}$$

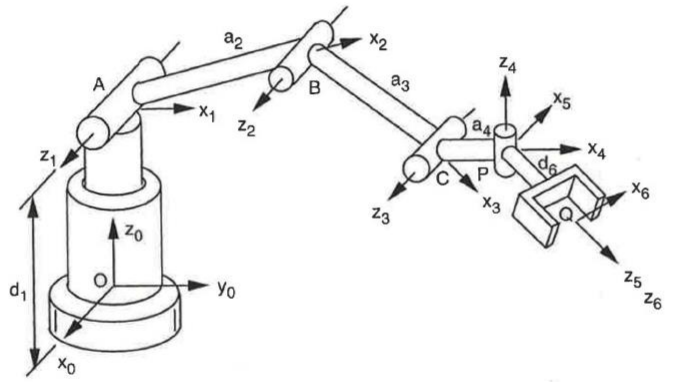
$$\Delta = a_1^2 + a_2^2 + 2a_1a_2c\theta_2$$

$$\theta_1 = \text{Atan2}(s\theta_1, c\theta_1).$$



Inverse Kinematics: Example II

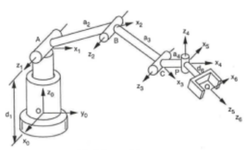
Elbow Manipulator:



Inverse Kinematics: Example II

- Target Position:

  $\mathbf{u} = [u_x, u_y, u_z]^T$, $\mathbf{v} = [v_x, v_y, v_z]^T$, $\mathbf{w} = [w_x, w_y, w_z]^T$, and $\mathbf{p} = [p_x, p_y, p_z]^T$.



- Transformation Matrices:

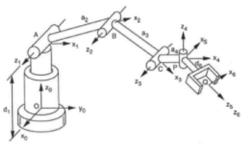
$$A_1 = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad ({}^0A_1)^{-1} = \begin{bmatrix} c\theta_1 & s\theta_1 & 0 & 0 \\ -s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} c\theta_2 & 0 & -s\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ s\theta_2 & 0 & c\theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} c\theta_3 & 0 & -s\theta_3 & a_2(1 - c\theta_3) \\ 0 & 1 & 0 & 0 \\ s\theta_3 & 0 & c\theta_3 & -a_2 s\theta_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_4 = \begin{bmatrix} c\theta_4 & 0 & -s\theta_4 & (a_2 + a_3)(1 - c\theta_4) \\ 0 & 1 & 0 & 0 \\ s\theta_4 & 0 & c\theta_4 & -(a_2 + a_3)s\theta_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_5 = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & (a_2 + a_3 + a_4)(1 - c\theta_5) \\ s\theta_5 & c\theta_5 & 0 & -(a_2 + a_3 + a_4)s\theta_5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse Kinematics: Example II

- Key Matrix Products:

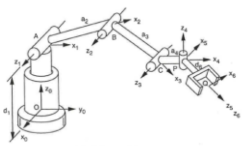


$$A_2A_3A_4 = \begin{bmatrix} c\theta_{234} & 0 & -s\theta_{234} & a_2c\theta_2 + a_3c\theta_{23} - (a_2+a_3)c\theta_{234} \\ 0 & 1 & 0 & 0 \\ s\theta_{234} & 0 & c\theta_{234} & a_2s\theta_2 + a_3s\theta_{23} - (a_2+a_3)s\theta_{234} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_1A_2A_3A_4 = \begin{bmatrix} c\theta_1c\theta_{234} & -s\theta_1 & -c\theta_1s\theta_{234} & c\theta_1[a_2c\theta_2 + a_3c\theta_{23} - (a_2+a_3)c\theta_{234}] \\ s\theta_1c\theta_{234} & c\theta_1 & -s\theta_1s\theta_{234} & s\theta_1[a_2c\theta_2 + a_3c\theta_{23} - (a_2+a_3)c\theta_{234}] \\ s\theta_{234} & 0 & c\theta_{234} & [a_2s\theta_2 + a_3s\theta_{23} - (a_2+a_3)s\theta_{234}] \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse Kinematics: Example II

- Inverse Kinematics:



$$\mathbf{p} = A_1A_2A_3A_4\mathbf{p}_0.$$

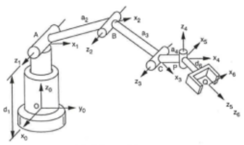
$$A_1^{-1}\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = A_2A_3A_4\begin{bmatrix} a_2+a_3+a_4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$p_xc\theta_1 + p_ys\theta_1 = a_2c\theta_2 + a_3c\theta_{23} + a_4c\theta_{234},$$

$$-p_xs\theta_1 + p_yc\theta_1 = 0,$$

$$p_z = a_2s\theta_2 + a_3s\theta_{23} + a_4s\theta_{234}.$$

Inverse Kinematics: Example II



- Solving the System:

$$\theta_1 = \tan^{-1}\frac{p_y}{p_x}.$$

$$\theta_5 = \sin^{-1}(-w_x s\theta_1 + w_y c\theta_1).$$

$$\theta_{234} = \mathrm{Atan2}\left[w_z/c\theta_5, (w_x c\theta_1 + w_y s\theta_1)/c\theta_5\right].$$

$$a_2 c\theta_2 + a_3 c\theta_{23} = k_1, \qquad k_1 = p_x c\theta_1 + p_y s\theta_1 - a_4 c\theta_{234}$$

$$a_2 s\theta_2 + a_3 s\theta_{23} = k_2, \qquad k_2 = p_z - a_4 s\theta_{234}$$

$$a_2^2 + a_3^2 + 2a_2a_3 c\theta_3 = k_1^2 + k_2^2.$$

$$\theta_3 = \cos^{-1}\frac{k_1^2 + k_2^2 - a_2^2 - a_3^2}{2a_2a_3}.$$

$$\theta_6 = \mathrm{Atan2}(s\theta_6, c\theta_6).$$