
DESIGN OF LINEAR STATE FEEDBACK CONTROL LAWS

Previous chapters, by introducing fundamental state-space concepts and analysis tools, have now set the stage for our initial foray into state-space methods for control system design. In this chapter, our focus is on the design of state feedback control laws that yield desirable closed-loop performance in terms of both transient and steady-state response characteristics. The fundamental result that underpins much of this chapter is that controllability of the open-loop state equation is both necessary and sufficient to achieve arbitrary closed-loop eigenvalue placement via state feedback. Furthermore, explicit feedback gain formulas for eigenvalue placement are available in the single-input case. To support the design of state feedback control laws, we discuss important relationship between the eigenvalue locations of a linear state equation and its dynamic response characteristics.

For state equations that are not controllable and so arbitrary eigenvalue placement via state feedback is not possible, we investigate whether state feedback can be used to at least stabilize the closed-loop state equation. This leads to the concept of stabilizability. Following that, we discuss techniques for improving steady-state performance first by introducing an additional gain parameter in the state feedback law, followed by the incorporation of integral-error compensation into our state feedback structure.

The chapter concludes by illustrating the use of MATLAB for shaping the dynamic response and state feedback control law design in the context of our Continuing MATLAB Example and Continuing Examples 1 and 2.

7.1 STATE FEEDBACK CONTROL LAW

We begin this section with the linear time-invariant state equation

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\end{aligned}\quad (7.1)$$

which represents the open-loop system or *plant* to be controlled. Our focus is on the application of state feedback control laws of the form

$$u(t) = -Kx(t) + r(t) \quad (7.2)$$

with the goal of achieving desired performance characteristics for the closed-loop state equation

$$\begin{aligned}\dot{x}(t) &= (A - BK)x(t) + Br(t) \\ y(t) &= Cx(t)\end{aligned}\quad (7.3)$$

The effect of state feedback on the open-loop block diagram of Figure 1.1 is shown in Figure 7.1.

The state feedback control law (7.2) features a constant state feedback gain matrix K of dimension $m \times n$ and a new external reference input $r(t)$ necessarily having the same dimension $m \times 1$ as the open-loop input $u(t)$, as well as the same physical units. Later in this chapter we will modify

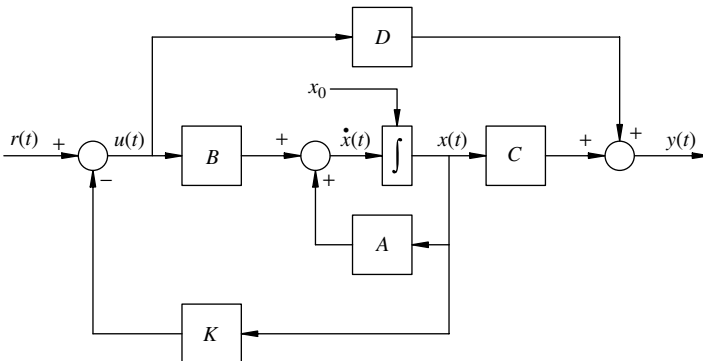


FIGURE 7.1 Closed-loop system block diagram.

the state feedback control law to include a gain matrix multiplying the reference input. The state feedback control law can be written in terms of scalar components as

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix} = - \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{m1} & k_{m2} & \cdots & k_{mn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} r_1(t) \\ r_2(t) \\ \vdots \\ r_m(t) \end{bmatrix}$$

For the single-input, single-output case, the feedback gain K is a $1 \times n$ row vector, the reference input $r(t)$ is a scalar signal, and the state feedback control law has the form

$$\begin{aligned} u(t) &= - [k_1 \quad k_2 \quad \cdots \quad k_n] \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + r(t) \\ &= -k_1x_1(t) - k_2x_2(t) - \cdots - k_nx_n(t) + r(t) \end{aligned}$$

If the external reference input is absent, the state feedback control law is called a *regulator* that is designed to deliver desirable transient response for nonzero initial conditions and/or attenuate disturbances to maintain the equilibrium state $\tilde{x} = 0$.

7.2 SHAPING THE DYNAMIC RESPONSE

In addition to closed-loop asymptotic stability (see Chapter 6), which requires that the closed-loop system dynamics matrix $A - BK$ have strictly negative real-part eigenvalues, we are often interested in other characteristics of the closed-loop transient response, such as rise time, peak time, percent overshoot, and settling time of the step response. Before we investigate the extent to which state feedback can influence the closed-loop eigenvalues, we first review topics associated with transient response performance of feedback control systems that are typically introduced in an undergraduate course emphasizing classical control theory. In our state-space context, we seek to translate desired transient response characteristics into specifications on system eigenvalues, which are closely related to transfer function poles. Specifying desired closed-loop system behavior via eigenvalue selection is called *shaping* the dynamic response. Control system engineers often use dominant first- and second-order

subsystems as approximations in the design process, along with criteria that justify such approximations for higher-order systems. Our discussion follows this approach.

Eigenvalue Selection for First-Order Systems

Figure 7.2 shows unit step responses of typical of first- through fourth-order systems.

For a first-order system, we can achieve desired transient behavior via specifying a single eigenvalue. Figure 7.2 (*top left*) shows a standard first-order system step response. All stable first-order systems driven by unit step inputs behave this way, with transient response governed by a single decaying exponential involving the time constant τ . After three time constants, the first-order unit step response is within 95 percent of its steady-state value. A smaller time constant responds more quickly, whereas a larger time constant responds more slowly. On specifying a desired time constant, the associated characteristic polynomial and eigenvalue are

$$\lambda + \frac{1}{\tau} \quad \text{and} \quad \lambda_1 = -\frac{1}{\tau}$$

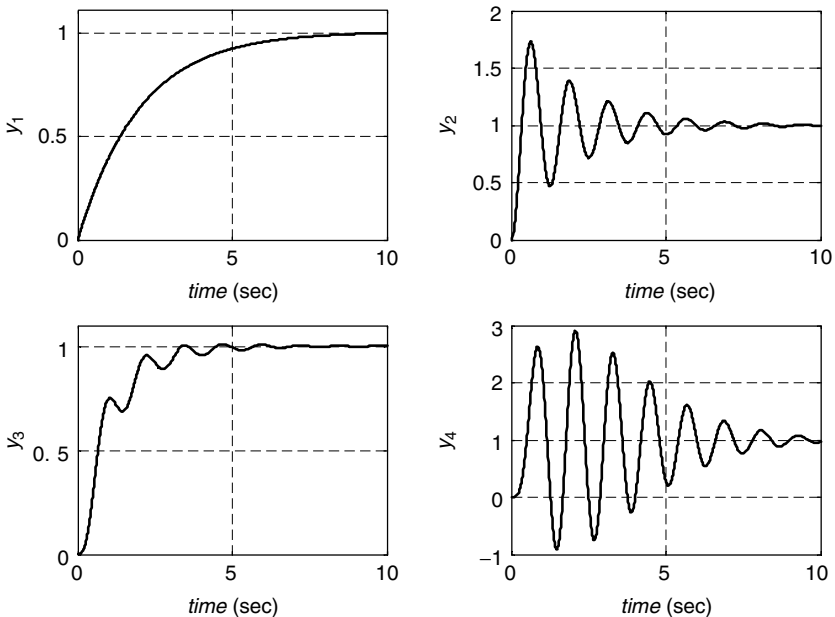


FIGURE 7.2 First- through fourth-order system unit step responses.

Eigenvalue Selection for Second-Order Systems

For a second-order system, we can achieve desired transient behavior via specifying a pair of eigenvalues. To illustrate, we consider the linear translational mechanical system of Example 1.1 (see Figure 1.2) with applied force $f(t)$ as the input and mass displacement $y(t)$ as the output. We identify this with a standard second-order system by redefining the input via $u(t) = f(t)/k$. The new input $u(t)$ can be interpreted as a commanded displacement. This will normalize steady-state value of the unit step response to 1.0. With this change, the state equation becomes

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{k}{m} \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

with associated transfer function

$$H(s) = \frac{\frac{k}{m}}{s^2 + \frac{c}{m}s + \frac{k}{m}}$$

We compare this with the standard second-order transfer function, namely,

$$\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

in which ξ is the unitless damping ratio, and ω_n is the undamped natural frequency in radians per second. This leads to the relationships

$$\xi = \frac{c}{2\sqrt{km}} \quad \text{and} \quad \omega_n = \sqrt{\frac{k}{m}}$$

The characteristic polynomial is

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = \lambda^2 + 2\xi\omega_n\lambda + \omega_n^2$$

from which the eigenvalues are

$$\lambda_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

TABLE 7.1 Damping Ratio versus Step Response Characteristics

| Case | Damping Ratio | Eigenvalues | Unit Step Response |
|-------------------|---------------|--------------------------------------|--|
| Overdamped | $\xi > 1$ | Real and distinct | Slowest transient response |
| Critically damped | $\xi = 1$ | Real and equal | Fastest transient response without overshoot |
| Underdamped | $0 < \xi < 1$ | Complex conjugate pair | Faster transient response but with overshoot and oscillation |
| Undamped | $\xi = 0$ | Imaginary pair | Undamped oscillation |
| Unstable | $\xi < 0$ | At least one with positive real part | Unbounded response |

To study the relationship between these eigenvalues and system transient response, we identify five distinct cases in Table 7.1, determined by the dimensionless damping ratio ξ for a fixed undamped natural frequency ω_n .

We next relate step response characteristics to the system eigenvalues for the most interesting of these cases: the underdamped case characterized by $0 < \xi < 1$. In this case, the complex conjugate eigenvalues are given by

$$\lambda_{1,2} = -\xi\omega_n \pm j\omega_d$$

in which $\omega_d = \omega_n\sqrt{1 - \xi^2}$ is the damped natural frequency in radians per second. The unit step response for a standard second-order system in the underdamped case is

$$y(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1 - \xi^2}} \sin(\omega_d t + \theta)$$

in which the phase angle is given by $\theta = \cos^{-1}(\xi)$ and therefore is referred to as the *damping angle*. This response features a sinusoidal component governed by the damped natural frequency and damping angle that is damped by a decaying exponential envelope related to the negative real part of the eigenvalues. A response of this type is plotted in Figure 7.3.

For the underdamped case, there are four primary performance characteristics (see Figure 7.3) associated with the unit step response that either directly or approximately can be related to the damping ratio and undamped natural frequency. The performance characteristics definitions and formulas are found in Dorf and Bishop (2005). Rise time t_R is defined as the elapsed time between when the response first reaches 10 percent of

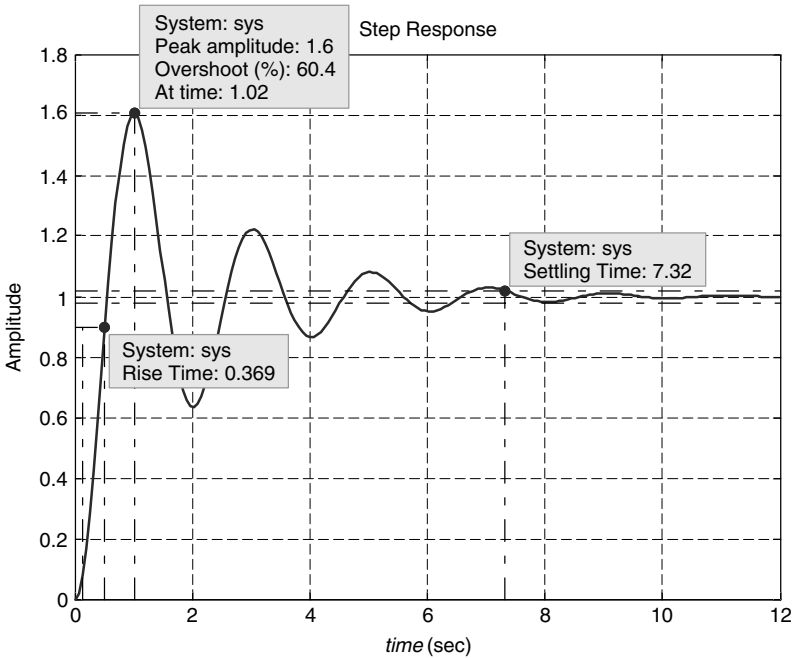


FIGURE 7.3 Unit step response for Example 7.1.

the steady-state value to when the response first reaches 90 percent of the steady-state value. For a damping ratio in the range $0.3 < \xi < 0.8$, rise time can be approximated by

$$t_R \cong \frac{2.16\xi + 0.60}{\omega_n}$$

Peak time t_p is the time at which the peak response value is reached and is given exactly by

$$t_p = \frac{\pi}{\omega_n \sqrt{1 - \xi^2}} = \frac{\pi}{\omega_d}$$

Percent overshoot PO characterizes the relationship between the peak value and steady-state value according to

$$PO = \frac{\text{peak value} - \text{steady-state value}}{\text{steady-state value}} \times 100\%$$

and can be computed exactly using

$$PO = 100e^{-\xi\pi/\sqrt{1-\xi^2}}$$

Settling time t_S is typically defined to be the time at which the response enters and remains within a ± 2 percent band about the steady-state value and can be approximated by

$$t_S \cong \frac{4}{\xi \omega_n}$$

The swiftness of the response is related to the rise time and peak time; the deviation between the response and its steady-state value is related to the percent overshoot and settling time.

Example 7.1 We return to the linear translational mechanical system for the parameter values $m = 1$ kg, $c = 1$ N-s/m, and $k = 10$ N/m. The undamped natural frequency and dimensionless damping ratio are

$$\begin{aligned}\omega_n &= \sqrt{\frac{k}{m}} = \sqrt{\frac{10}{1}} = 3.16 \text{ rad/s} \\ \xi &= \frac{c}{2\sqrt{km}} = \frac{1}{2\sqrt{10(1)}} = 0.158\end{aligned}$$

and the damped natural frequency is

$$\omega_d = \omega_n \sqrt{1 - \xi^2} = 3.12 \text{ rad/s}$$

The characteristic polynomial is

$$\lambda^2 + 2\xi\omega_n\lambda + \omega_n^2 = \lambda^2 + \lambda + 10$$

yielding the complex conjugate eigenvalues

$$\lambda_{1,2} = -0.5 \pm 3.12i$$

The unit step response is given by

$$y(t) = 1 - 1.01e^{-0.5t} \sin(3.12t + 80.9^\circ)$$

We calculate directly from ξ and ω_n the step response characteristics: $t_R = 0.30$ s (because $\xi < 0.3$ this estimate may be somewhat inaccurate), $t_P = 1.01$ s, $PO = 60.5$ percent, and $t_S = 8$ s.

A plot of the unit step response is given in Figure 7.3, with rise time, peak time, percent overshoot, and settling time displayed. This plot was obtained from MATLAB using the **step** command and right-clicking in the

resulting figure window to add the performance specifications (Characteristics → choose Peak Response, Settling Time, Rise Time, and Steady State in turn. MATLAB will mark the response with a dot, which the user may click to display the numerical values). We see that with the exception of rise time (0.30 s estimated versus 0.37 s from the plot), the formula values agree well with the MATLAB results values labeled on Figure 7.3 (settling time is also inaccurate, 8 s from the equation versus 7.32 s as measured from the MATLAB generated response). □

Example 7.2 The unit step response in Figure 7.3 is typical of lightly damped systems such as large-scale space structures. We consider the application of a proportional-derivative (PD) control law given by

$$\begin{aligned} u(t) &= -k_P y(t) - k_D \dot{y}(t) + Gr(t) \\ &= - \begin{bmatrix} k_P & k_D \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + Gr(t) \end{aligned}$$

which, as shown, also has the form of a state feedback control law. The closed-loop system that results is given by

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\left(\frac{k}{m} + k_P\right) & -\left(\frac{c}{m} + k_D\right) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{k}{m} \end{bmatrix} Gr(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{aligned}$$

Our objective is to improve the step response characteristics by reducing the percent overshoot and settling time via adjusting the proportional and derivative gains k_P and k_D . In particular, we specify a desired percent overshoot of 4 percent and a settling time of 2 s. The gain G will be chosen so that the closed-loop unit step response will have the same steady-state value as the open-loop unit step response.

In terms of these performance specifications, we calculate the desired closed-loop damping ratio ξ' from the percent-overshoot formula according to

$$\xi' = \frac{\left| \ln \left(\frac{PO}{100} \right) \right|}{\sqrt{\pi^2 + \left[\ln \left(\frac{PO}{100} \right) \right]^2}} = \frac{\left| \ln \left(\frac{4}{100} \right) \right|}{\sqrt{\pi^2 + \left[\ln \left(\frac{4}{100} \right) \right]^2}} = 0.716$$

Using this, we calculate the desired closed-loop undamped natural frequency from the settling-time formula via

$$\omega'_n = \frac{4}{\xi' t_S} = 2.79 \text{ rad/s}$$

The desired closed-loop damped natural frequency is

$$\omega'_d = \omega'_n \sqrt{1 - (\xi')^2} = 1.95 \text{ rad/s}$$

The proportional and derivative gains then can be determined by comparing the closed-loop characteristic polynomial of the closed-loop state equation with the desired closed-loop characteristic polynomial:

$$\lambda^2 + (2\xi\omega_n + k_D)\lambda + (\omega_n^2 + k_P) = \lambda^2 + 2\xi'\omega'_n\lambda + \omega_n'^2 = \lambda^2 + 4\lambda + 7.81$$

which leads to

$$k_P = \omega_n'^2 - \omega_n^2 = -2.20 \quad k_D = 2\xi'\omega'_n - 2\xi\omega_n = 3.00$$

The input gain G is determined from the relationship

$$G = \frac{\omega_n'^2}{\omega_n^2} = 0.781$$

which yields the closed-loop transfer function

$$\begin{aligned} \frac{\frac{k}{m}G}{s^2 + (\frac{c}{m} + k_D)s + (\frac{k}{m} + k_P)} &= \frac{\omega_n'^2 G}{s^2 + (2\xi\omega_n + k_D)s + (\omega_n^2 + k_P)} \\ &= \frac{\omega_n'^2}{s^2 + 2\xi'\omega'_n s + \omega_n'^2} = \frac{7.81}{s^2 + 4s + 7.81} \end{aligned}$$

The closed-loop eigenvalues then are:

$$\lambda_{1,2} = -2 \pm j1.95$$

and the closed-loop unit step response is

$$y(t) = 1 - 1.43e^{-2t} \sin(1.95t + 44.3^\circ)$$

The MATLAB **step** command gives us the four performance specifications: $t_R = 0.78$ s, $t_P = 1.60$ s, $PO = 4$ percent, and $t_S = 2.12$ s. The actual settling time differs slightly from the desired settling time because of the approximation in the settling-time formula. Figure 7.4 shows a

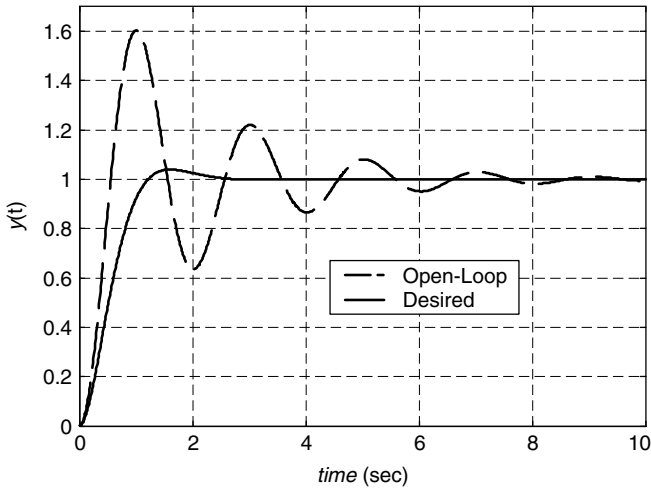


FIGURE 7.4 Closed-loop versus open-loop system unit step responses for Example 7.2.

comparison of the unit step response for the improved closed-loop system with the unit step response of the open-loop system from Figure 7.3.

In Figure 7.4 and Table 7.2 we see that the step response of the closed-loop system has a slower rise and peak time compared with the open-loop system, but the percent overshoot and settling time are much improved. Values given below for rise time and settling time are from MATLAB-generated step responses rather than the approximate formulas. \square

Inequality Constraints When specifying desired transient response behavior, the control system engineer typically specifies bounds on the step response characteristics rather than exact values. For first-order systems, this usually involves an upper bound on the time required for the step response to reach steady state, which yields an upper bound for the desired time constant. For second-order systems, this typically involves

TABLE 7.2 Comparison of Open-Loop and Closed-Loop Step Response Characteristics

| Specification | Open-Loop Response | Closed-Loop Response |
|---------------|--------------------|----------------------|
| t_R (sec) | 0.37 | 0.78 |
| t_P (sec) | 1.02 | 1.60 |
| PO (%) | 60.4 | 4 |
| t_S (sec) | 7.32 | 2.12 |

any combination of upper bounds the on rise time, peak time, percent overshoot, and settling time of the step response. This yields bounds on the desired damping ratio and undamped natural frequency that, in turn, can be translated into regions in the complex plane that specify desired eigenvalue locations. This more general viewpoint will be demonstrated in the following example for the second-order case.

Example 7.3 In this example we characterize acceptable eigenvalue locations in the complex plane so that the following second-order performance specifications are satisfied:

$$PO \leq 4\% \quad t_S \leq 2 \text{ s} \quad t_P \leq 0.5 \text{ s}$$

The formula for percent overshoot indicates that percent overshoot decreases as damping ratio increases. Therefore, an upper bound on percent overshoot $PO \leq PO_{\max}$ corresponds to a lower bound on damping ratio $\xi \geq \xi_{\min}$. From Example 7.2, we know that a damping ratio of $\xi_{\min} = 0.716$ will yield a percent overshoot equal to $PO_{\max} = 4$ percent. Therefore, in order to achieve a percent overshoot less than 4 percent, we require a damping ratio greater than 0.716. We previously defined the damping angle $\theta = \cos^{-1}(\xi)$, which appeared in the expression for the unit step response of the standard underdamped second-order system. From this relationship, we see that a lower bound on the damping ratio corresponds to an upper bound on the damping angle $\theta \leq \theta_{\max} = \cos^{-1}(\xi_{\min})$. To relate this to allowable eigenvalue locations, we see from basic trigonometry that

$$\tan(\theta) = \frac{\sqrt{1 - \xi^2}}{\xi} = \frac{\omega_n \sqrt{1 - \xi^2}}{\xi \omega_n} = \frac{\omega_d}{\xi \omega_n}$$

Recalling that eigenvalues of an underdamped second-order system are $\lambda_{1,2} = -\xi \omega_n \pm j \omega_d$, in which $\omega_d = \omega_n \sqrt{1 - \xi^2}$ is the damped natural frequency, we therefore interpret the damping angle θ as the angle made by the radial line from the origin in the complex plane passing through the eigenvalue $\lambda_1 = -\xi \omega_n + j \omega_d$ measured with respect to the negative real axis. By symmetry, $-\theta$ is the angle made by the radial line from the origin in the complex plane passing through the conjugate eigenvalue $\lambda_2 = -\xi \omega_n - j \omega_d$, also measured with respect to the negative real axis. An upper bound on the damping angle characterizes a sector in the left half of the complex plane bounded by the pair of radial lines corresponding to the maximum allowable damping angle. In this example, a minimum allowable damping ratio of 0.716 corresponds to a maximum allowable damping angle of $\cos^{-1}(0.716) = 44.3^\circ$. The associated sector specifying

acceptable eigenvalue locations is bounded by the radial lines making angles of $\pm 44.3^\circ$ with respect to the negative real axis.

The formula that approximates settling time shows that settling time is inversely proportional to the product $\xi\omega_n$ that is directly related to the real part of the complex-conjugate eigenvalues. Therefore, an upper bound on settling time $t_S \leq t_{S, \max}$ corresponds to eigenvalues that lie to the left of a vertical line passing through the point $-4/t_{S, \max}$ on the negative real axis. With a specified upper bound on settling time of 2 s, eigenvalues must lie to the left of the vertical line passing through the point $-4/2 = -2$ on the negative real axis.

The formula for peak time shows that peak time is inversely proportional to the damped natural frequency ω_d that characterizes the imaginary part of the complex-conjugate eigenvalues. Thus an upper bound on peak time $t_P \leq t_{P, \max}$ yields a lower bound on ω_d . This corresponds to an eigenvalue that lies above the horizontal line passing through $j(\pi/t_{P, \max})$ on the positive imaginary axis and a conjugate eigenvalue that lies below the horizontal line passing through $-j(\pi/t_{P, \max})$ on the negative imaginary axis. With a specified upper bound on peak time of 0.5 s, these horizontal lines pass through $\pm j(\pi/0.5) = \pm j2\pi$.

The region in the complex plane characterizing allowable eigenvalue locations with respect to all three constraints is given by the intersection of the individual regions. The shaded region in Figure 7.5 shows allowable eigenvalue locations for this example along with the bounding lines for the individual regions.

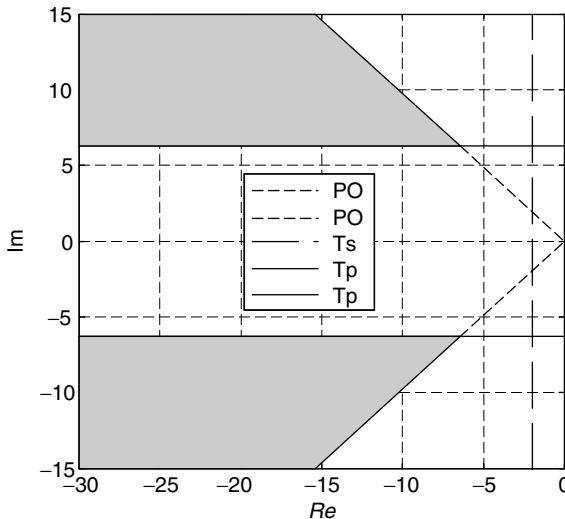


FIGURE 7.5 Allowable eigenvalue locations for Example 7.3.

In this example we see that only the percent-overshoot and peak-time constraints are active. That is, the settling time will be less than 2 s for all eigenvalues satisfying the other two constraints. □

Higher-Order Systems

We typically encounter systems of order or dimension greater than two. When we approximate higher-order systems by a dominant first- or second-order model and specify the associated dominant eigenvalue locations that meet various transient response specifications, we eventually need to specify locations for the remaining eigenvalues. The general rule of thumb we adopt is to augment the dominant eigenvalues with the requisite number of additional eigenvalues that are 10 times further to the left than the dominant eigenvalues in the complex plane. In so doing, we expect that the higher-order system response will be dominated by that of the first- or second-order approximate model.

Figure 7.6 shows the effect of augmenting a dominant first-order system eigenvalue ($\lambda_1 = -0.5$) with additional real negative eigenvalues at least 10 times higher, for second- through fourth-order systems. The eigenvalues associated with each case are given in Table 7.3. The second- through fourth-order step responses are similar to the desired dominant first-order step response. The settling time increases slightly as the system order increases.

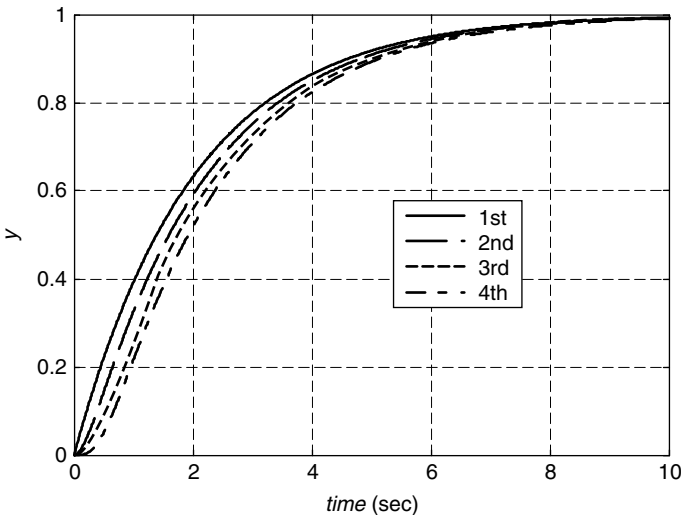
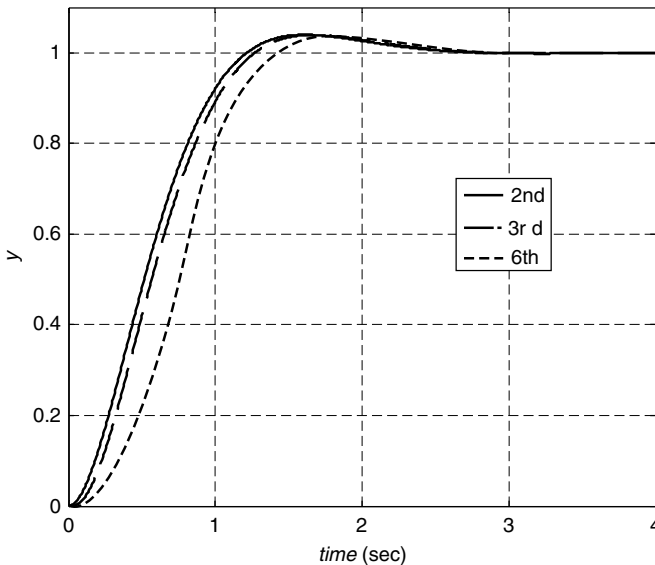


FIGURE 7.6 Dominant first-order system versus second- through fourth-order systems.

TABLE 7.3 Eigenvalues for Figure 7.6

| System Order | Eigenvalues |
|--------------|--|
| First | $\lambda_1 = -0.5$ |
| Second | $\lambda_{1,2} = -0.5, -5$ |
| Third | $\lambda_{1,2,3} = -0.5, -5, -6$ |
| Fourth | $\lambda_{1,2,3,4} = -0.5, -5, -6, -7$ |

Figure 7.7 shows the effect of augmenting dominant second-order system eigenvalues ($s_{1,2} = -2 \pm 1.95i$ from Example 7.2) with additional real negative eigenvalues at least 10 times further to the left in the complex plane for third- and sixth-order systems. The eigenvalues associated with each case are given in Table 7.4. The third- and sixth-order step responses are similar to the desired dominant second-order step response. The rise time increases slightly as the system order increases.

**FIGURE 7.7** Dominant second-order system versus third- and sixth-order systems.**TABLE 7.4 Eigenvalues for Figure 7.7**

| System Order | Eigenvalues |
|--------------|--|
| Second | $\lambda_{1,2} = -2 \pm 1.95i$ |
| Third | $\lambda_{1,2,3} = -2 \pm 1.95i, -20$ |
| Sixth | $\lambda_{1,2,3,4,5,6} = -2 \pm 1.95i, -20, -21, -22, -23$ |

ITAE Method for Shaping the Dynamic Response

The ITAE (integral of time multiplying the absolute value of error), method attempts to accomplish dynamic shaping by penalizing the error, in our case the deviation between the unit step response and the steady-state value that yields the transient response component, more heavily later in the response as time increases (Graham and Lathrop, 1953). The ITAE objective function is

$$\text{ITAE} = \int_0^{\infty} t|e(t)|dt$$

Minimizing the ITAE objective function yields a step response with relatively small overshoot and relatively little oscillation. The ITAE is just one possible objective function; others have been proposed and used in practice. For first- through sixth-order systems, the characteristic polynomials given in Table 7.5 correspond to systems that minimize the ITAE criterion (Dorf and Bishop, 2005). In each case, one must specify the desired natural frequency ω_n (higher values correspond to faster response). Then the desired eigenvalues are obtained by finding the roots of the appropriate characteristic polynomial for the particular system order.

Figure 7.8 shows the unit step responses for the ITAE first- through sixth-order systems described by transfer functions of the form

$$H_k(s) = \frac{\omega_n^k}{d_k(s)}$$

in which the index k denotes the system order, and the denominator polynomial is taken from Table 7.5. The independent axis in Figure 7.8 is the normalized time $\omega_n t$, which is unitless. Note that beyond the first-order case, some overshoot is involved in optimizing the ITAE objective function.

This section has presented several approaches to translate transient response specifications into desired eigenvalue locations. These methods

TABLE 7.5 ITAE Characteristic Polynomials

| System Order | Characteristic Polynomial |
|--------------|--|
| First | $s + \omega_n$ |
| Second | $s^2 + 1.4 \omega_n s + \omega_n^2$ |
| Third | $s^3 + 1.75 \omega_n s^2 + 2.15 \omega_n^2 s + \omega_n^3$ |
| Fourth | $s^4 + 2.1 \omega_n s^3 + 3.4 \omega_n^2 s^2 + 2.7 \omega_n^3 s + \omega_n^4$ |
| Fifth | $s^5 + 2.8 \omega_n s^4 + 5.0 \omega_n^2 s^3 + 5.5 \omega_n^3 s^2 + 3.4 \omega_n^4 s + \omega_n^5$ |
| Sixth | $s^6 + 3.25 \omega_n s^5 + 6.6 \omega_n^2 s^4 + 8.6 \omega_n^3 s^3 + 7.45 \omega_n^4 s^2 + 3.95 \omega_n^5 s + \omega_n^6$ |

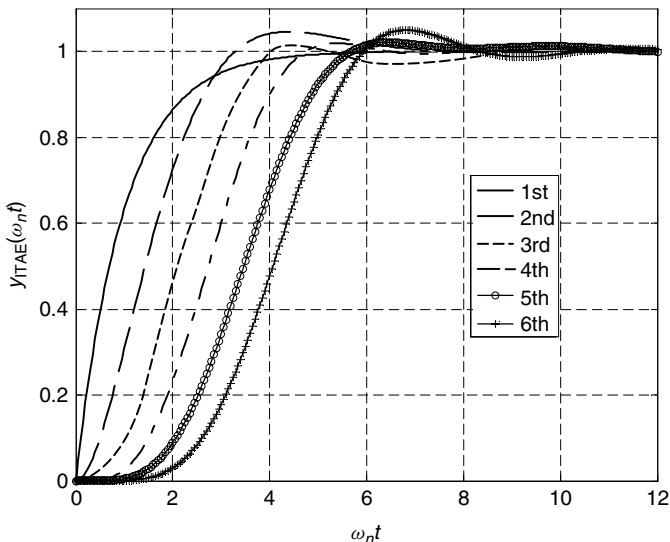


FIGURE 7.8 ITAE unit step responses.

have focused on so-called all-pole system models. It is well known from classical control that the presence of zeros (roots of the transfer function numerator polynomial) will change the system's transient behavior. One approach to remedy this is to use a prefilter to reduce the effect of these zeros.

7.3 CLOSED-LOOP EIGENVALUE PLACEMENT VIA STATE FEEDBACK

The following result establishes a connection between the ability to arbitrarily place the closed-loop eigenvalues by proper choice of the state feedback gain matrix K and controllability of the open-loop state equation, i.e., the pair (A, B) .

Theorem 7.1 *For any symmetric set of n complex numbers $\{\mu_1, \mu_2, \dots, \mu_n\}$, there exists a state feedback gain matrix K such that $\sigma(A - BK) = \{\mu_1, \mu_2, \dots, \mu_n\}$ if and only if the pair (A, B) is controllable.*

Proof. Here we prove that controllability of the pair (A, B) is necessary for arbitrary eigenvalue placement via state feedback. That controllability of the pair (A, B) is sufficient for arbitrary eigenvalue placement via state feedback will be established constructively in the single-input case by deriving formulas for the requisite state feedback gain vector. The

multiple-input case is considerably more complicated, and we will be content to refer the interested reader to treatments available in the literature.

For necessity, we prove the contrapositive. Suppose the pair (A, B) is not controllable so that

$$\text{rank} [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B] = q < n$$

and there exists a nonsingular $n \times n$ matrix T for which

$$\begin{aligned} \hat{A} &= T^{-1}AT & \hat{B} &= T^{-1}B \\ &= \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} & &= \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \end{aligned}$$

where A_{11} is of dimension $q \times q$ and B_1 is $q \times m$. For any state feedback gain matrix K , let

$$\begin{aligned} \hat{K} &= KT \\ &= [K_1 \quad K_2] \end{aligned}$$

where K_1 is of dimension $m \times q$ and K_2 is $m \times (n - q)$. Then

$$\begin{aligned} \hat{A} - \hat{B}\hat{K} &= \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} - \begin{bmatrix} B_1 \\ 0 \end{bmatrix} [K_1 \quad K_2] \\ &= \begin{bmatrix} A_{11} - B_1K_1 & A_{12} - B_1K_2 \\ 0 & A_{22} \end{bmatrix} \end{aligned}$$

in which we note that the bottom-block row is unaffected by the state feedback. The characteristic polynomial is

$$\begin{aligned} |sI - (\hat{A} - \hat{B}\hat{K})| &= \begin{vmatrix} sI - (A_{11} - B_1K_1) & -(A_{12} - B_1K_2) \\ 0 & sI - A_{22} \end{vmatrix} \\ &= |sI - (A_{11} - B_1K_1)| \cdot |sI - A_{22}| \end{aligned}$$

Thus the n eigenvalues of $\hat{A} - \hat{B}\hat{K}$ are the q eigenvalues of $A_{11} - B_1K_1$ along with the $n - q$ eigenvalues of A_{22} , which represents the uncontrollable subsystem. Since

$$\begin{aligned} \hat{A} - \hat{B}\hat{K} &= T^{-1}AT - T^{-1}BKT \\ &= T^{-1}(A - BK)T \end{aligned}$$

the same conclusion holds for the eigenvalues of $A - BK$. Since for any feedback gain matrix K the eigenvalues of A_{22} are always among the closed-loop eigenvalues, we conclude that arbitrary eigenvalue placement is *not* possible. Equivalently, if arbitrary eigenvalue placement is possible by state feedback, then the pair (A, B) necessarily must be controllable. \square

Feedback Gain Formula for Controller Canonical Form

We know from Chapter 3 that the controller canonical form specifies a controllable single-input state equation. The construction of a feedback gain vector that achieves arbitrary eigenvalue placement proceeds as follows: The coefficient matrices for controller canonical form are given below:

$$A_{CCF} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \quad B_{CCF} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

and we recall that since A_{CCF} is a companion matrix, its characteristic polynomial is written down by inspection:

$$|sI - A_{CCF}| = s^n + a_{n-1}s^{n-1} + \cdots + a_2s^2 + a_1s + a_0$$

We denote a feedback gain vector by

$$K_{CCF} = [\delta_0 \quad \delta_1 \quad \delta_2 \quad \cdots \quad \delta_{n-1}]$$

which yields the closed-loop system dynamics matrix

$$A_{CCF} - B_{CCF}K_{CCF} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 - \delta_0 & -a_1 - \delta_1 & -a_2 - \delta_2 & \cdots & -a_{n-1} - \delta_{n-1} \end{bmatrix}$$

We see that $A_{CCF} - B_{CCF}K_{CCF}$ is also in companion form, so its characteristic polynomial can be written down by inspection of the bottom row:

$$|sI - A_{CCF} + B_{CCF}K_{CCF}| = s^n + (a_{n-1} + \delta_{n-1})s^{n-1} + \cdots + (a_2 + \delta_2)s^2 + (a_1 + \delta_1)s + (a_0 + \delta_0)$$

Beginning with an arbitrary symmetric set of complex numbers $\{\mu_1, \mu_2, \dots, \mu_n\}$ that represents the desired closed-loop eigenvalues, we define the associated closed-loop characteristic polynomial

$$\begin{aligned}\alpha(s) &= (s - \mu_1)(s - \mu_2) \cdots (s - \mu_n) \\ &= s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_2s^2 + \alpha_1s + \alpha_0\end{aligned}$$

It is important to note that because the roots of a polynomial uniquely determine and are uniquely determined by the polynomial coefficients, specifying n desired closed-loop eigenvalues is equivalent to specifying the n coefficients $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_{n-1}$. In terms of these parameters, the crux of the problem is to determine K_{CCF} so that the characteristic polynomial of $A_{CCF} - B_{CCF}K_{CCF}$ matches the desired closed-loop characteristic polynomial $\alpha(s)$. Equating

$$\begin{aligned}|sI - A_{CCF} + B_{CCF}K_{CCF}| &= s^n + (a_{n-1} + \delta_{n-1})s^{n-2} + \cdots + (a_2 + \delta_2)s^2 \\ &\quad + (a_1 + \delta_1)s + (a_0 + \delta_0) \\ &= s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_2s^2 + \alpha_1s + \alpha_0\end{aligned}$$

yields, on comparing coefficients of like powers of s , the relationships

$$a_0 + \delta_0 = \alpha_0 \quad a_1 + \delta_1 = \alpha_1 \quad a_2 + \delta_2 = \alpha_2 \quad \cdots \quad a_{n-1} + \delta_{n-1} = \alpha_{n-1}$$

so that

$$\delta_0 = \alpha_0 - a_0 \quad \delta_1 = \alpha_1 - a_1 \quad \delta_2 = \alpha_2 - a_2 \quad \cdots \quad \delta_{n-1} = \alpha_{n-1} - a_{n-1}$$

and the state feedback gain vector is then given by

$$K_{CCF} = [(\alpha_0 - a_0) \quad (\alpha_1 - a_1) \quad (\alpha_2 - a_2) \quad \cdots \quad (\alpha_{n-1} - a_{n-1})]$$

Example 7.4 We consider the following three-dimensional state equation given in controller canonical form specified by the coefficient matrices

$$A_{CCF} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -18 & -15 & -2 \end{bmatrix} \quad B_{CCF} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C_{CCF} = [1 \quad 0 \quad 0]$$

The open-loop characteristic polynomial, is by inspection,

$$a(s) = s^3 + a_2s^2 + a_1s + a_0 = s^3 + 2s^2 + 15s + 18$$

which yields the open-loop eigenvalues $\lambda_{1,2,3} = -1.28, -0.36 \pm j3.73$. This open-loop system exhibits a typical third-order lightly damped step response, as shown in Figure 7.9 below.

This open-loop system is already asymptotically stable, but we are interested in designing a state feedback control law to improve the transient response performance. We first specify a pair of dominant eigenvalues to yield a percent overshoot of 6 percent and a settling time of 3 s. The associated damping ratio and undamped natural frequency are $\xi = 0.67$ and $\omega_n = 2.00$ rad/s. The resulting dominant second-order eigenvalues are $\lambda_{1,2} = -1.33 \pm j1.49$. The open-loop system is third-order, so we need to specify a third desired eigenvalue, which we choose to be negative, real, and 10 further to the left of the dominant second-order eigenvalues in the complex plane: $\lambda_3 = -13.33$. Thus the desired characteristic polynomial is

$$\alpha(s) = s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0 = s^3 + 16s^2 + 39.55s + 53.26$$

This leads immediately to the state feedback gain vector

$$\begin{aligned} K_{CCF} &= [(\alpha_0 - a_0) \quad (\alpha_1 - a_1) \quad (\alpha_2 - a_2)] \\ &= [(53.26 - 18) \quad (39.55 - 15) \quad (16 - 2)] \\ &= [35.26 \quad 24.55 \quad 14.00] \end{aligned}$$

The state feedback control law

$$u(t) = -K_{CCF} x_{CCF}(t) + r(t)$$

yields the closed-loop state equation specified by the coefficient matrices

$$\begin{aligned} A_{CCF} - B_{CCF}K_{CCF} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -53.26 & -39.55 & -16 \end{bmatrix} & B_{CCF} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ C_{CCF} &= [1 \quad 0 \quad 0] \end{aligned}$$

which is also in controller canonical form. Figure 7.9 shows a comparison of the open-loop and closed-loop output responses to a unit step input. The closed-loop transient response is greatly improved, and the achieved percent overshoot (5.9 percent) and settling time (3.09 s) are in close agreement with the design specifications. Note, however, that neither unit step response achieves a steady-state value of 1.0. For the closed-loop system, this means that the steady-state output does not match the reference

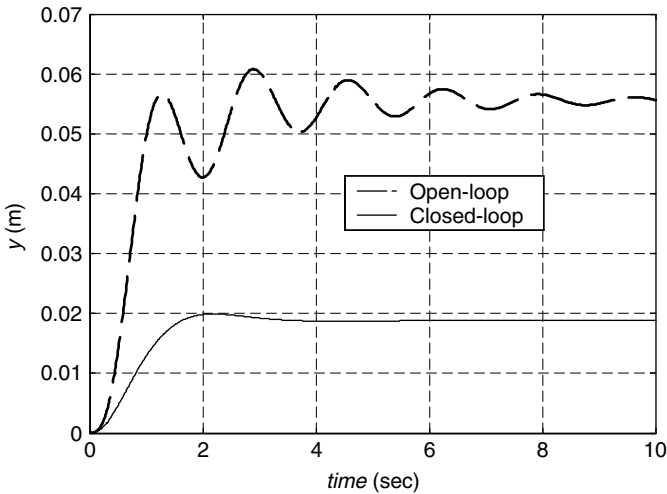


FIGURE 7.9 Open-loop versus closed-loop unit step responses for Example 7.4.

input, yielding, in this case, substantial steady-state error. Methods to correct this will be addressed later in this chapter. \square

Bass-Gura Formula

We generalize the preceding feedback gain construction for the controller canonical form to the case of an arbitrary controllable single-input state equation. Our plan of attack is to use the explicitly defined state coordinate transformation linking a controllable single-input state equation to its associated controller canonical form.

As before, we let $\{\mu_1, \mu_2, \dots, \mu_n\}$ be a symmetric set of n complex numbers representing the desired closed-loop eigenvalues, which uniquely determines the desired closed-loop characteristic polynomial

$$\begin{aligned}\alpha(s) &= (s - \mu_1)(s - \mu_2) \cdots (s - \mu_n) \\ &= s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_2s^2 + \alpha_1s + \alpha_0\end{aligned}$$

For the controllable pair (A, B) , Section 3.4 indicates that the state coordinate transformation $x(t) = T_{\text{CCF}} x_{\text{CCF}}(t)$ with

$$T_{\text{CCF}} = P P_{\text{CCF}}^{-1}$$

transforms the original state equation to its controller canonical form. Since the system dynamics matrices A and A_{CCF} are related by a similarity

transformation, their characteristic polynomials are the same, that is,

$$|sI - A| = |sI - A_{CCF}| = s^n + a_{n-1}s^{n-2} + \cdots + a_2s^2 + a_1s + a_0$$

With

$$K_{CCF} = [(\alpha_0 - a_0) \quad (\alpha_1 - a_1) \quad (\alpha_2 - a_2) \quad \cdots \quad (\alpha_{n-1} - a_{n-1})]$$

achieving the desired eigenvalue placement for $A_{CCF} - B_{CCF}K_{CCF}$, we set $K = K_{CCF}T_{CCF}^{-1}$, from which $K_{CCF} = KT_{CCF}$ and

$$\begin{aligned} A_{CCF} - B_{CCF}K_{CCF} &= T_{CCF}^{-1}AT_{CCF} - T_{CCF}^{-1}B(KT_{CCF}) \\ &= T_{CCF}^{-1}(A - BK)T_{CCF} \end{aligned}$$

Thus

$$\begin{aligned} |sI - A + BK| &= |sI - A_{CCF} + B_{CCF}K_{CCF}| \\ &= s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_2s^2 + \alpha_1s + \alpha_0 \end{aligned}$$

so that $K = K_{CCF}T_{CCF}^{-1}$ achieves the desired eigenvalue placement for the closed-loop state equation in the original coordinates.

Putting everything together yields the *Bass-Gura formula*, namely,

$$\begin{aligned} K &= K_{CCF}T_{CCF}^{-1} \\ &= [(\alpha_0 - a_0) \quad (\alpha_1 - a_1) \quad (\alpha_2 - a_2) \quad \cdots \quad (\alpha_{n-1} - a_{n-1})] [PP_{CCF}^{-1}]^{-1} \\ &= [(\alpha_0 - a_0) \quad (\alpha_1 - a_1) \quad (\alpha_2 - a_2) \quad \cdots \quad (\alpha_{n-1} - a_{n-1})] \\ &\quad \times \left[P \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & 1 \\ a_2 & a_3 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \right]^{-1} \end{aligned} \tag{7.4}$$

which depends only on the controllable open-loop pair (A, B) , the open-loop characteristic polynomial, and the desired closed-loop characteristic polynomial, as determined by the desired closed-loop eigenvalues. We observe that nonsingularity of the controllability matrix P is required to implement this formula. Finally, we note that by providing a formula for the state feedback gain vector that achieves arbitrary eigenvalue placement for a controllable single-input state equation, we have proved the sufficiency part of Theorem 7.1, at least in the single-input case.

Example 7.5 We demonstrate application of the Bass-Gura formula for the three-dimensional single-input state equation with coefficient matrices

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

We take the desired closed-loop eigenvalues to be $\{\mu_1, \mu_2, \mu_3\} = \{-1, -1, -1\}$. We break the feedback gain vector computation into several steps.

We first check the controllability of the pair (A, B) by computing

$$\begin{aligned} P &= [B \quad AB \quad A^2B] \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \end{aligned}$$

which has $|P| = 2 \neq 0$, so this state equation is controllable. We note that the same conclusion follows by inspection because the state equation is in diagonal canonical form, in which the eigenvalues of A are distinct and the elements of B are all nonzero.

Next, we form the open-loop and desired closed-loop characteristic polynomials. Here, A is a diagonal matrix with its eigenvalues displayed on the main diagonal $\lambda_{1,2,3} = 0, 1, 2$. The open-loop characteristic polynomial and associated coefficients are

$$\begin{aligned} a(s) &= (s - 0)(s - 1)(s - 2) \\ &= s^3 - 3s^2 + 2s + 0 \end{aligned} \quad a_2 = -3, \quad a_1 = 2, \quad a_0 = 0$$

The open-loop system is unstable because there are two positive real eigenvalues. The closed-loop desired characteristic polynomial and associated coefficients are

$$\begin{aligned} \alpha(s) &= (s + 1)^3 \\ &= s^3 + 3s^2 + 3s + 1 \end{aligned} \quad \alpha_2 = 3, \quad \alpha_1 = 3, \quad \alpha_0 = 1$$

From this we can directly compute

$$\begin{aligned} K_{CCF} &= [(\alpha_0 - a_0) \quad (\alpha_1 - a_1) \quad (\alpha_2 - a_2)] \\ &= [(1 - 0) \quad (3 - 2) \quad (3 - (-3))] \\ &= [1 \quad 1 \quad 6] \end{aligned}$$

We next calculate T_{CCF} and T_{CCF}^{-1} :

$$\begin{aligned} T_{CCF} &= P P_{CCF}^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & -3 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad T_{CCF}^{-1} = \begin{bmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & -1 & 1 \\ 0 & -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -3 & 1 \\ 0 & -2 & 1 \\ 0 & -1 & 1 \end{bmatrix} \end{aligned}$$

Finally, we calculate the state feedback gain vector

$$\begin{aligned} K &= K_{CCF} T_{CCF}^{-1} \\ &= [1 \quad 1 \quad 6] \begin{bmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & -1 & 1 \\ 0 & -1 & 2 \end{bmatrix} \\ &= \left[\frac{1}{2} \quad -8 \quad \frac{27}{2} \right] \end{aligned}$$

As a final check, we see that

$$A - BK = \begin{bmatrix} -\frac{1}{2} & 8 & -\frac{27}{2} \\ -\frac{1}{2} & 9 & -\frac{27}{2} \\ -\frac{1}{2} & 8 & -\frac{23}{2} \end{bmatrix}$$

has eigenvalues $\{-1, -1, -1\}$ as desired.

Figure 7.10 compares the MATLAB generated state variable step responses for open and closed-loop state equations each for zero initial state. Because the open-loop state equation is in diagonal canonical form, we can easily compute

$$x_1(t) = t \quad x_2(t) = e^t - 1 \quad x_3(t) = \frac{1}{2}(e^{2t} - 1)$$

which grow without bound as time increases. The stabilizing effect of the state feedback law is evident from the closed-loop responses. \square

Ackermann's Formula

Here we present *Ackermann's formula* for computing the state feedback gain vector. In terms of the desired closed-loop characteristic polynomial

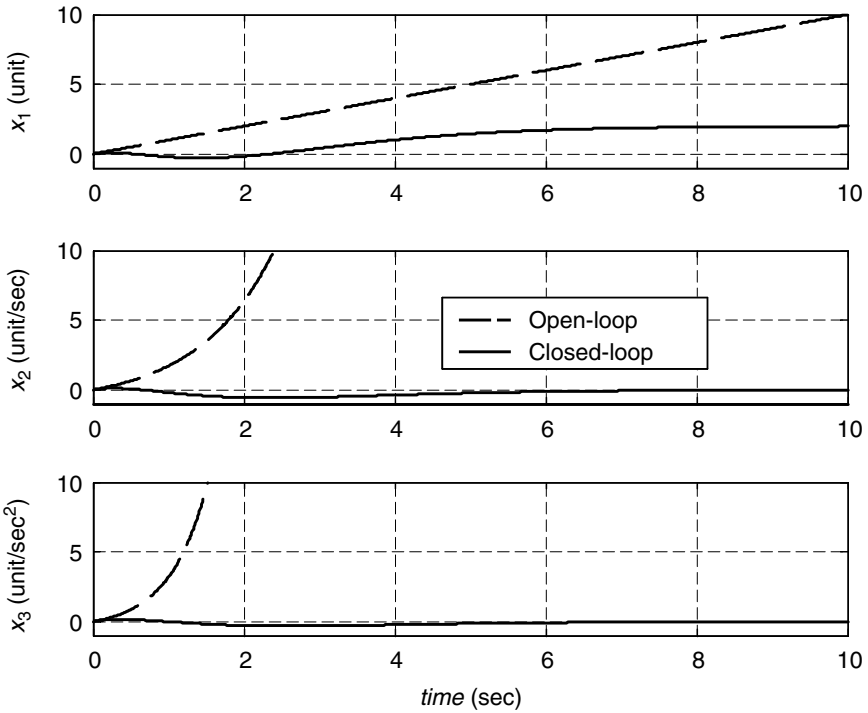


FIGURE 7.10 Open-loop versus closed-loop step response for Example 7.5.

$\alpha(s)$, the state feedback gain vector is given by

$$K = [0 \ 0 \ \dots \ 0 \ 1] P^{-1} \alpha(A)$$

Here $P = [B \ AB \ A^2B \ \dots \ A^{n-1}B]$ is the controllability matrix for the controllable pair (A, B) with inverse P^{-1} and $\alpha(A)$ represents

$$\alpha(A) = A^n + \alpha_{n-1}A^{n-1} + \dots + \alpha_2A^2 + \alpha_1A + \alpha_0I$$

which yields an $n \times n$ matrix. To verify Ackermann’s formula, it is again convenient to begin with a controllable pair in controller canonical form, for which we must check that

$$\begin{aligned} K_{CCF} &= [(\alpha_0 - a_0) \ (\alpha_1 - a_1) \ (\alpha_2 - a_2) \ \dots \ (\alpha_{n-1} - a_{n-1})] \\ &= [0 \ 0 \ \dots \ 0 \ 1] P_{CCF}^{-1} \alpha(A_{CCF}) \end{aligned}$$

We first observe that the special structure of P_{CCF}^{-1} leads to

$$\begin{aligned} [0 \ 0 \ \cdots \ 0 \ 1] P_{CCF}^{-1} &= [0 \ 0 \ \cdots \ 0 \ 1] \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & 1 \\ a_2 & a_3 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \\ &= [1 \ 0 \ \cdots \ 0 \ 0] \end{aligned}$$

Now, with $a(s)$ denoting the characteristic polynomial of A_{CCF} , the Cayley-Hamilton theorem gives

$$\begin{aligned} a(A_{CCF}) &= A_{CCF}^n + a_{n-1}A_{CCF}^{n-1} + \cdots + a_2A_{CCF}^2 + a_1A_{CCF} + a_0I \\ &= 0 \quad (n \times n) \end{aligned}$$

which allows us to write

$$\begin{aligned} \alpha(A_{CCF}) &= \alpha(A_{CCF}) - a(A_{CCF}) \\ &= (A_{CCF}^n + \alpha_{n-1}A_{CCF}^{n-1} + \cdots + \alpha_2A_{CCF}^2 + \alpha_1A_{CCF} + \alpha_0I) \\ &\quad - (A_{CCF}^n + a_{n-1}A_{CCF}^{n-1} + \cdots + a_2A_{CCF}^2 + a_1A_{CCF} + a_0I) \\ &= (\alpha_{n-1} - a_{n-1})A_{CCF}^{n-1} + \cdots + (\alpha_2 - a_2)A_{CCF}^2 \\ &\quad + (\alpha_1 - a_1)A_{CCF} + (\alpha_0 - a_0)I \end{aligned}$$

Next, the special structure of A_{CCF} yields

$$\begin{aligned} [1 \ 0 \ 0 \ \cdots \ 0 \ 0] I &= [1 \ 0 \ 0 \ \cdots \ 0 \ 0] \\ [1 \ 0 \ 0 \ \cdots \ 0 \ 0] A_{CCF} &= [0 \ 1 \ 0 \ \cdots \ 0 \ 0] \\ [1 \ 0 \ 0 \ \cdots \ 0 \ 0] A_{CCF}^2 &= [0 \ 1 \ 0 \ \cdots \ 0 \ 0] A_{CCF} \\ &= [0 \ 0 \ 1 \ \cdots \ 0 \ 0] \\ &\vdots \\ [1 \ 0 \ 0 \ \cdots \ 0 \ 0] A_{CCF}^{n-1} &= [0 \ 0 \ 0 \ \cdots \ 1 \ 0] A_{CCF}^{n-2} \\ &= [0 \ 0 \ 0 \ \cdots \ 0 \ 1] \end{aligned}$$

By combining these intermediate results, we see that

$$\begin{aligned}
& [0 \ 0 \ \cdots \ 0 \ 1] P_{\text{CCF}}^{-1} \alpha(A_{\text{CCF}}) \\
&= [1 \ 0 \ \cdots \ 0 \ 0] [(\alpha_{n-1} - a_{n-1}) A_{\text{CCF}}^{n-1} + \cdots + (\alpha_2 - a_2) A_{\text{CCF}}^2 \\
&\quad + (\alpha_1 - a_1) A_{\text{CCF}} + (\alpha_0 - a_0) I] \\
&= (\alpha_0 - a_0) [1 \ 0 \ \cdots \ 0 \ 0] I \\
&\quad + (\alpha_1 - a_1) [1 \ 0 \ \cdots \ 0 \ 0] A_{\text{CCF}} \\
&\quad + (\alpha_2 - a_2) [1 \ 0 \ \cdots \ 0 \ 0] A_{\text{CCF}}^2 \\
&\quad + \\
&\quad \vdots \\
&\quad + (\alpha_{n-1} - a_{n-1}) [1 \ 0 \ \cdots \ 0 \ 0] A_{\text{CCF}}^{n-1} \\
&= [(\alpha_0 - a_0) \ (\alpha_1 - a_1) \ (\alpha_2 - a_2) \ \cdots \ (\alpha_{n-1} - a_{n-1})] \\
&= K_{\text{CCF}}
\end{aligned}$$

as required. Now, to address the general case, we let (A, B) denote a controllable pair for which $T_{\text{CCF}} = P P_{\text{CCF}}^{-1}$ is such that $A_{\text{CCF}} = T_{\text{CCF}}^{-1} A T_{\text{CCF}}$. It is straightforward to verify that for any integer $k \geq 0$, $A_{\text{CCF}}^k = (T_{\text{CCF}}^{-1} A T_{\text{CCF}})^k = T_{\text{CCF}}^{-1} A^k T_{\text{CCF}}$, which allows us to write

$$\begin{aligned}
\alpha(A_{\text{CCF}}) &= A_{\text{CCF}}^n + \alpha_{n-1} A_{\text{CCF}}^{n-1} + \cdots + \alpha_2 A_{\text{CCF}}^2 + \alpha_1 A_{\text{CCF}} + \alpha_0 I \\
&= (T_{\text{CCF}}^{-1} A^n T_{\text{CCF}}) + \alpha_{n-1} (T_{\text{CCF}}^{-1} A^{n-1} T_{\text{CCF}}) \\
&\quad + \cdots + \alpha_2 (T_{\text{CCF}}^{-1} A^2 T_{\text{CCF}}) + \alpha_1 (T_{\text{CCF}}^{-1} A T_{\text{CCF}}) + \alpha_0 (T_{\text{CCF}}^{-1} T_{\text{CCF}}) \\
&= T_{\text{CCF}}^{-1} (A^n + \alpha_{n-1} A^{n-1} + \cdots + \alpha_2 A^2 + \alpha_1 A + \alpha_0 I) T_{\text{CCF}} \\
&= T_{\text{CCF}}^{-1} \alpha(A) T_{\text{CCF}}
\end{aligned}$$

We then have

$$\begin{aligned}
K &= K_{\text{CCF}} T_{\text{CCF}}^{-1} \\
&= [0 \ 0 \ \cdots \ 0 \ 1] P_{\text{CCF}}^{-1} \alpha(A_{\text{CCF}}) T_{\text{CCF}}^{-1} \\
&= [0 \ 0 \ \cdots \ 0 \ 1] P_{\text{CCF}}^{-1} (T_{\text{CCF}}^{-1} \alpha(A) T_{\text{CCF}}) T_{\text{CCF}}^{-1} \\
&= [0 \ 0 \ \cdots \ 0 \ 1] P_{\text{CCF}}^{-1} T_{\text{CCF}}^{-1} \alpha(A)
\end{aligned}$$

$$\begin{aligned}
&= [0 \ 0 \ \cdots \ 0 \ 1] P_{\text{CCF}}^{-1} (P P_{\text{CCF}}^{-1})^{-1} \alpha(A) \\
&= [0 \ 0 \ \cdots \ 0 \ 1] P^{-1} \alpha(A)
\end{aligned}$$

as desired.

Example 7.6 To demonstrate the use of Ackermann's formula, we now repeat the feedback gain computation first considered in Example 7.4 for the state equation in controller canonical form specified by

$$A_{\text{CCF}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -18 & -15 & -2 \end{bmatrix} \quad B_{\text{CCF}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

for which, by inspection,

$$P_{\text{CCF}}^{-1} = \begin{bmatrix} 15 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

For the desired closed-loop characteristic polynomial

$$\alpha(s) = s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0 = s^3 + 16s^2 + 39.55s + 53.26$$

we compute, with the aid of MATLAB,

$$\begin{aligned}
\alpha(A_{\text{CCF}}) &= A_{\text{CCF}}^3 + 16A_{\text{CCF}}^2 + 39.55A_{\text{CCF}} + 53.26I \\
&= \begin{bmatrix} 35.26 & 24.55 & 14.00 \\ -252.00 & -174.74 & -3.45 \\ 62.10 & -200.25 & -167.84 \end{bmatrix}
\end{aligned}$$

Ackermann's formula then gives

$$\begin{aligned}
K &= [0 \ 0 \ 1] P_{\text{CCF}}^{-1} \alpha(A_{\text{CCF}}) \\
&= [1 \ 0 \ 0] \begin{bmatrix} 35.26 & 24.55 & 14.00 \\ -252.00 & -174.74 & -3.45 \\ 62.10 & -200.25 & -167.84 \end{bmatrix} \\
&= [35.26 \ 24.55 \ 14.00]
\end{aligned}$$

This result agrees with K_{CCF} as computed in Example 7.4. This is so because for controllable single-input state equations, the feedback gain vector is uniquely determined by the desired closed-loop eigenvalues. \square

Multiple-Input, Multiple-Output Eigenvalue Placement

One approach to addressing eigenvalue placement for controllable multiple-input state equations is to first derive a multiple-input version of our controller canonical form. This is considerably more complicated in the multiple-input case, but once this has been accomplished, the subsequent construction of a state feedback gain matrix that achieves desired closed-loop eigenvalues follows in a straightforward fashion. We do not pursue the details here and instead refer the interested reader to Chapters 13 and 14 in Rugh (1996) and Chapters 6 and 7 in Kailath (1980). We are reassured by the fact that for computational purposes, the MATLAB **place** function can accommodate multiple-input state equations (see Section 7.6).

7.4 STABILIZABILITY

We have seen that for a controllable state equation, arbitrary closed-loop eigenvalue placement can be achieved using state feedback. Since the freedom to assign closed-loop eigenvalues implies that we can asymptotically stabilize the closed-loop system via state feedback, we can say that controllability is a *sufficient* condition for asymptotic stabilization via state feedback. In situations where the plant is not controllable, it is natural to ask whether or not asymptotic stabilization via state feedback is still possible. We answer this question by introducing the concept of *stabilizability*. It turns out that we have already developed the key ingredients for the requisite analysis in our previous investigations of controllability and state feedback. To motivate the ensuing discussion, we consider the following example. The state equation

$$\dot{x}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = [1 \ 0 \ 0] x(t)$$

is already in the standard form for an uncontrollable state equation, as can be seen from the partitioning

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & -1 & 1 \\ \hline 0 & 0 & -2 \end{array} \right] \quad \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

in which the pair (A_{11}, B_1) specifies a controllable two-dimensional subsystem. In terms of a state feedback gain vector $K = [k_1 \ k_2 \ k_3]$, we have

$$\begin{aligned} A - BK &= \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & -1 & 1 \\ \hline 0 & 0 & -2 \end{array} \right] - \left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array} \right] [k_1 \ k_2 \ | \ k_3] \\ &= \left[\begin{array}{cc|c} \left[\begin{array}{cc} 1 & 0 \\ 1 & -1 \end{array} \right] - \left[\begin{array}{c} 1 \\ 1 \end{array} \right] [k_1 \ k_2] & & \left[\begin{array}{c} 0 \\ 1 \end{array} \right] - \left[\begin{array}{c} 1 \\ 1 \end{array} \right] k_3 \\ \hline 0 & 0 & -2 \end{array} \right] \end{aligned}$$

The gains k_1 and k_2 can be chosen to arbitrarily locate the eigenvalues of the controllable subsystem. Because of the upper block triangular structure, the three eigenvalues of $A - BK$ are the two freely assigned by proper choice of k_1 and k_2 together with the third at -2 contributed by the uncontrollable subsystem. Thus we conclude that even though the state equation is not controllable, and hence *arbitrary* eigenvalue placement via state feedback is not achievable, it is still possible to construct a state feedback gain vector such that $A - BK$ specifies an asymptotically stable closed-loop state equation. This is a direct result of the fact that the uncontrollable subsystem, which cannot be influenced by state feedback, is itself asymptotically stable. We also see that the gain k_3 has no influence on the eigenvalues of $A - BK$. We now formalize this notion in the following definition.

Definition 7.2 *The linear state equation (7.1) [or the pair (A, B) , for short] is stabilizable if there exists a state feedback gain matrix K for which all eigenvalues of $A - BK$ have strictly negative real part.*

Note that as a consequence of this definition and our prior eigenvalue placement results, we see that controllability implies stabilizability. On the other hand, the preceding example indicates that the converse does not hold. A state equation can be stabilizable but not controllable. We therefore regard stabilizability as a weaker condition than controllability.

The preceding example provides hints for general characterizations of stabilizability. If the pair (A, B) is controllable, then it is stabilizable as noted above and so we consider the case in which the pair (A, B) is not controllable. We know that there exists a coordinate transformation $x(t) = Tz(t)$ such that the transformed state equation has

$$\hat{A} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \quad \hat{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

in which (A_{11}, B_1) is a controllable pair. With $\hat{K} = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$ a conformably partitioned state feedback gain matrix, we have

$$\hat{A} - \hat{B}\hat{K} = \begin{bmatrix} A_{11} - B_1K_1 & A_{12} - B_1K_2 \\ 0 & A_{22} \end{bmatrix}$$

whose eigenvalues are those of $A_{11} - B_1K_1$, along with those of the A_{22} . Because (A_{11}, B_1) is a controllable pair, K_1 can be chosen such that the eigenvalues of $A_{11} - B_1K_1$ have negative real part. However, the uncontrollable subsystem associated with A_{22} is uninfluenced by state feedback, and the eigenvalues of A_{22} will be among the eigenvalues of $\hat{A} - \hat{B}\hat{K}$ for any \hat{K} . Consequently, in order for every eigenvalue of $\hat{A} - \hat{B}\hat{K}$ to have negative real parts, the eigenvalues of A_{22} must have negative real parts to begin with. We therefore see that in the case where the pair (A, B) is not controllable, stabilizability requires that the uncontrollable subsystem be asymptotically stable. We also observe that K_2 plays no role in this analysis. Our next step is to adapt our previously derived Popov-Belevitch-Hautus tests for controllability in order to provide algebraic criteria for stabilizability that do not require a particular coordinate transformation.

Theorem 7.3 *The following statements are equivalent:*

1. *The pair (A, B) is stabilizable.*
2. *There exists no left eigenvector of A associated with an eigenvalue having nonnegative real part that is orthogonal to the columns of B ;*
3. *The matrix $\begin{bmatrix} \lambda I - A & B \end{bmatrix}$ has full row-rank for all complex λ with nonnegative real part.*

Proof. To establish the equivalence of these statements, it is enough to prove the chain of implications: 1 implies 2, 2 implies 3, and 3 implies 1.

For the first implication, we prove the contrapositive: If there does exist a left eigenvector of A associated with an eigenvalue having nonnegative real part that is orthogonal to the columns of B , then the pair (A, B) is not stabilizable. Suppose that for $\lambda \in \sigma(A)$ with $\text{Re}(\lambda) \geq 0$ there exists $w \in \mathbb{C}^n$ that satisfies

$$w \neq 0 \quad w^*A = \lambda w^* \quad w^*B = 0$$

Then, for any state feedback gain matrix K , we have

$$\begin{aligned} w^*(A - BK) &= w^*A - (w^*B)K \\ &= \lambda w^* \end{aligned}$$

which indicates that λ is also an eigenvalue of $A - BK$ with non-negative real part. Thus, by definition, the pair (A, B) is not stabilizable.

We next prove the contrapositive of the second implication: If $[\lambda I - A \quad B]$ has less than full-row rank for some complex λ with nonnegative real part, then there does exist a left eigenvector of A associated with an eigenvalue having nonnegative real part that is orthogonal to the columns of B . Suppose that $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) \geq 0$ is such that the matrix $[\lambda I - A \quad B]$ has linearly dependent rows. Consequently, there exists a nontrivial linear combination of the rows of $[\lambda I - A \quad B]$ yielding a $1 \times (n + m)$ zero vector. This is equivalent to the existence of a nonzero vector $w \in \mathbb{C}^n$ that satisfies

$$w^* [\lambda I - A \quad B] = [0 \quad 0]$$

which can be reorganized as

$$w^* A = \lambda w^* \quad w^* B = 0$$

Thus λ with $\text{Re}(\lambda) \geq 0$ is an eigenvalue of A with associated left eigenvector w that is orthogonal to the columns of B .

Finally, we prove the contrapositive of the third implication: If the pair (A, B) is not stabilizable, then there exists $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) \geq 0$ for which the matrix $[\lambda I - A \quad B]$ has less than full-row rank. If the pair (A, B) is not stabilizable, then it is not controllable, and a change of coordinates yields the transformed state equation with

$$\hat{A} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \quad \hat{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

in which the pair (A_{11}, B_1) is controllable. If it is not possible to find a stabilizing feedback gain matrix, then it must be the case that A_{22} has an eigenvalue $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) \geq 0$. By definition, $\lambda I - A_{22}$ is singular and therefore has linearly dependent rows, implying the same for

$$[\lambda I - \hat{A} \quad \hat{B}] = \begin{bmatrix} \lambda I - A_{11} & -A_{12} & B_1 \\ 0 & \lambda I - A_{22} & 0 \end{bmatrix}$$

Finally, the identity

$$[\lambda I - A \quad B] = T [\lambda I - \hat{A} \quad \hat{B}] \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix}$$

together with the fact that matrix rank is unaffected by pre- and postmultiplication by nonsingular matrices, implies that $[\lambda I - A \quad B]$ has less than full row rank for this $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) \geq 0$. \square

Example 7.7 Recall the uncontrollable state equation introduced in Example 3.3 and revisited in Examples 3.5 and 3.6:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} u(t)$$

We saw in Example 3.5 that the coordinate transformation $x(t) = Tz(t)$ with

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -3 & 0 \\ -3 & 7 & 1 \end{bmatrix}$$

yields the transformed state equation with coefficient matrices

$$\begin{aligned} \hat{A} &= T^{-1}AT & \hat{B} &= T^{-1}B \\ &= \left[\begin{array}{cc|c} 0 & -2 & 1 \\ 1 & -3 & 0 \\ \hline 0 & 0 & -3 \end{array} \right] & &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

which is in the standard form for an uncontrollable state equation. Moreover, the uncontrollable subsystem is one-dimensional with $A_{22} = -3$. Hence this state equation is stabilizable. Arbitrarily choosing two closed-loop eigenvalues to be $-2 + j2$ and $-2 - j2$, the Bass-Gura formula applied to the controllable pair (A_{11}, B_1) extracted from the pair (\hat{A}, \hat{B}) yields

$$K_1 = [1 \quad 3]$$

for which

$$A_{11} - B_1 K_1 = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1 \quad 3] = \begin{bmatrix} -1 & -5 \\ 1 & -3 \end{bmatrix}$$

has these specified eigenvalues. Therefore

$$\hat{K} = [1 \quad 3 \quad 0]$$

is such that $\hat{A} - \hat{B}\hat{K}$ has the three eigenvalues $-2 + j2$, $-2 - j2$, and -3 . Finally, the feedback gain vector

$$K = \hat{K}T^{-1} = [1 \quad 3 \quad 0] \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 3 & 1 \end{bmatrix} = [6 \quad 1 \quad 0]$$

yields $A - BK$ also having closed-loop eigenvalues $-2 + j2$, $-2 - j2$, and -3 .

We also saw in Example 3.6 that $w_3 = [2 \ 3 \ 1]^T$ is a left eigenvector of A associated with the eigenvalue $\lambda_3 = -3 < 0$ for which $w_3^T B = 0$. We can make the stronger assertion that the only left eigenvectors of A orthogonal to B are nonzero scalar multiples of w_3 and therefore are also associated with λ_3 . We conclude from Theorem 7.3 that since there exists no left eigenvector of A associated with an eigenvalue having nonnegative real part that is orthogonal to B , the state equation is stabilizable. We also see that $\lambda_3 = -3$ is the only value of λ for which $[\lambda I - A \ B]$ has less than rank 3, so $[\lambda I - A \ B]$ has full-row rank 3 for all λ with nonnegative real part, and again, stabilizability follows from Theorem 7.3. \square

7.5 STEADY-STATE TRACKING

We have studied in detail the extent to which state feedback control laws can influence the transient response characteristics by virtue of our complete freedom to specify closed-loop eigenvalues for a controllable state equation and our understanding of how eigenvalue locations affect the transient response. In this section we address a steady-state tracking requirement for step reference inputs. The first approach we consider involves a modification of our state feedback control law to include a gain that multiplies the reference input. This idea has been implemented already, albeit informally, in Example 7.2. Our second approach revisits in our state-space framework a familiar concept from classical control: Zero steady-state tracking error for step reference inputs can be achieved for so-called type 1 systems that include integral action on the tracking error. We refer to such control systems as *servomechanisms*.

Input Gain

As noted earlier, we now consider state feedback control laws of the form

$$u(t) = -Kx(t) + Gr(t)$$

in which the reference input is now multiplied by a gain G to be chosen so that for a step reference input $r(t) = R$, $t \geq 0$, the output of the closed-loop state equation

$$\begin{aligned}\dot{x}(t) &= (A - BK)x(t) + BGr(t) \\ y(t) &= Cx(t)\end{aligned}$$

satisfies

$$y_{\text{ss}} \triangleq \lim_{t \rightarrow \infty} y(t) = R$$

We note at the outset that this steady-state tracking requirement makes sense only when the output $y(t)$ and the reference input $r(t)$ have the same dimension p in the multiple-input, multiple-output case. This, in turn, dictates that the gain matrix G has dimension $m \times p$. In the single-input, single-output case, $y(t)$, $r(t)$, and $u(t)$ are scalar signals, and G is a scalar gain. The ensuing analysis is cast in the multi-input, multi-output case for which we will encounter the restriction $m \geq p$; i.e., the open-loop state equation has at least as many inputs as outputs. Clearly, the single-input, single-output case meets this requirement. We also assume from this point on that the feedback gain K has been chosen to yield an asymptotically stable closed-loop state equation.

From a frequency-domain viewpoint, the steady-state tracking objective requires that the closed-loop transfer function

$$H_{\text{CL}}(s) = C(sI - A + BK)^{-1}BG$$

have what we refer to as identity dc gain; that is, $H_{\text{CL}}(0) = I$ ($p \times p$). For then, with $R(s) = R(1/s)$ we may apply the final-value theorem (because the closed-loop state equation is asymptotically stable) to obtain

$$y_{\text{ss}} = \lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s) = \lim_{s \rightarrow 0} sH_{\text{CL}}(s)R\frac{1}{s} = H_{\text{CL}}(0)R$$

and so $y_{\text{ss}} = R$ for any constant vector $R \in \mathbb{R}^p$. The closed-loop dc gain is given by

$$H_{\text{CL}}(0) = -C(A - BK)^{-1}BG$$

in which $A - BK$ is guaranteed to be nonsingular because it is assumed to have strictly negative real-part eigenvalues. The question now is whether the gain matrix G can be chosen to yield $H_{\text{CL}}(0) = I$. We first consider the case $m = p$; i.e., the open-loop state equation has as many inputs as outputs. In this case, the dc gain is the product of the $p \times p$ matrix $-C(A - BK)^{-1}B$ and the $p \times p$ input gain matrix G . If the first factor is nonsingular, we can achieve identity dc gain by setting

$$G = -[C(A - BK)^{-1}B]^{-1}$$

If $-C(A - BK)^{-1}B$ is singular, the closed-loop state equation is said to have a *transmission zero* at $s = 0$. In the single-input, single-output case,

$-C(A - BK)^{-1}B$ is a scalar quantity and is therefore singular when and only when it is zero, meaning that the scalar transfer function $C(sI - A + BK)^{-1}B$ has a zero at $s = 0$ in the familiar sense.

For the case $m \geq p$, the factor $-C(A - BK)^{-1}B$ has dimension $p \times m$ and therefore has more columns than rows. If this matrix has full-row rank p then we can take the $m \times p$ input gain matrix G to be the *Moore-Penrose pseudoinverse* given by

$$G = -[C(A - BK)^{-1}B]^T [C(A - BK)^{-1}B[C(A - BK)^{-1}B]^T]^{-1}$$

which the reader should verify yields $H_{CL}(0) = I$.

We can arrive at the same conclusion from a time-domain viewpoint. For the constant reference input $r(t) = R$, $t \geq 0$, steady state corresponds to an equilibrium condition for the closed-loop state equation involving an equilibrium state denoted by x_{ss} that satisfies

$$0 = (A - BK)x_{ss} + BG R$$

which can be solved to give $x_{ss} = -(A - BK)^{-1}BG R$. The steady-state output then is obtained from

$$y_{ss} = Cx_{ss} = -C(A - BK)^{-1}BG R = H_{CL}(0)R$$

from which the same formula for the input gain results.

The preceding analysis can be generalized to address other steady-state objectives involving a closed-loop dc gain other than an identity matrix. For the $m = p$ case, let K_{dc} denote the desired closed-loop dc gain. We need only adjust the input gain formula according to

$$G = -[C(A - BK)^{-1}B]^{-1}K_{dc}$$

to achieve the desired result.

Example 7.8 We modify the state feedback control law computed for the state equation of Example 7.4 to include an input gain chosen so that the open-loop and closed-loop unit step responses reach the same steady-state value. The open-loop state equation is specified by the following coefficient matrices which although are in controller canonical form, we omit the corresponding subscripts

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -18 & -15 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = [1 \quad 0 \quad 0]$$

The open-loop transfer function is, by inspection,

$$H(s) = \frac{1}{s^3 + 2s^2 + 15s + 18}$$

from which the open-loop DC gain is $H(0) = 1/18 = 0.056$.

In Example 7.4, the state feedback gain was computed to be

$$K = [35.26 \quad 24.55 \quad 14.00]$$

to yield closed-loop eigenvalues at $\lambda_{1,2} = -1.33 \pm j1.49$, -13.33 in order to achieve a percent overshoot of 6 percent and a settling time of 3 s for the closed-loop unit step response. Without an input gain correction, the closed-loop transfer function is

$$C(sI - A + BK)^{-1}B = \frac{1}{s^3 + 16s^2 + 39.55s + 53.26}$$

which again is determined by inspection from the closed-loop state equation, which is also in controller canonical form. From this we compute $-C(A - BK)^{-1}B = 1/53.26 = 0.0188$. Clearly, unity closed-loop dc gain is achieved for $G = 53.26$, and the closed-loop dc gain will match the open-loop dc gain for

$$G = (53.26) \frac{1}{18} = 2.96$$

This input gain is used to produce the closed-loop unit step response plotted in of Figure 7.11. The closed-loop unit step response exhibits the improved transient response for which the feedback gain vector K was constructed, as well as a steady-state value that matches the open-loop unit step response by virtue of our input gain selection. \square

Servomechanism Design

Using an input gain to influence closed-loop steady-state performance requires accurate knowledge of the open-loop state equation's coefficient matrices in order to invert $-C(A - BK)^{-1}B$. In practice, model uncertainty, parameter variations, or intentional approximations can result in deviations between the nominal coefficient matrices used for the input gain computation and the actual system description. This has the potential to significantly alter the actual steady-state behavior.

Here we present a servomechanism design methodology (Ogata, 2002) that combines the classical approach of adding an integral-error term to

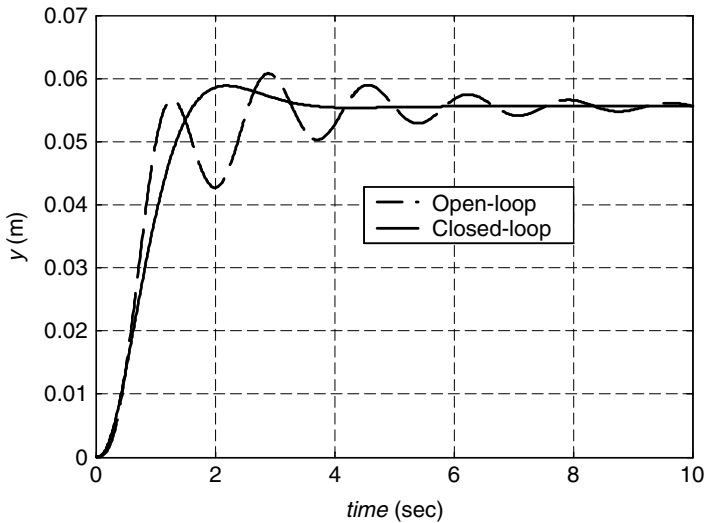


FIGURE 7.11 Open-loop versus closed-loop unit step response for Example 7.8.

obtain a type I system that yields zero steady-state tracking error for step reference inputs with our methods for state feedback design that deliver closed-loop stability and desirable transient response characteristics. This approach is *robust* with respect to uncertainty in the open-loop state equation in that the steady-state tracking performance is preserved as long as closed-loop stability is maintained. In this section we focus on the single-input, single-output case and impose the following additional assumptions:

Assumptions

1. The open-loop state equation, i.e., the pair (A, B) , is controllable.
2. The open-loop state equation has no pole/eigenvalue at $s = 0$.
3. The open-loop state equation has no zero at $s = 0$.

Our control law will be of the form

$$\begin{aligned}\dot{\xi}(t) &= r(t) - y(t) \\ u(t) &= -Kx(t) + k_1\xi(t)\end{aligned}$$

in which $r(t)$ is the step reference input to be tracked by the output $y(t)$. By setting the time derivative $\dot{\xi}(t)$ to equal the tracking error $r(t) - y(t)$, we see that $\xi(t)$ represents the integral of the tracking error. Taking

Laplace transforms for zero initial condition $\xi(0^-) = 0$ gives

$$s\xi(s) = R(s) - Y(s) = E(s)$$

$$\xi(s) = \frac{E(s)}{s}$$

which indicates that the integral error term introduces an open-loop pole at $s = 0$. Assumption 2 is in place so that the transfer function associated with the open-loop state equation does not itself contribute a pole at $s = 0$, in which case the new state variable $\xi(t)$ is not required. Assumption 3 prevents the pole at $s = 0$ introduced by the control law from being canceled by a zero at $s = 0$. We thus are guaranteed that the integral error term in the control law yields a type I system. The remainder of the control law can be written as

$$u(t) = - \begin{bmatrix} K & -k_I \end{bmatrix} \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}$$

which we interpret as a state feedback law involving the *augmented* $(n + 1)$ -dimensional state vector consisting of the open-loop state vector $x(t)$ together with the integrator state variable $\xi(t)$. The associated closed-loop system block diagram is shown in Figure 7.12, where “A, B Plant” refers to the open-loop state differential equation in Equation (7.1).

The interconnection of the feedback control law and the open-loop plant yields the $(n + 1)$ -dimensional closed-loop state equation:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} A - BK & Bk_I \\ -C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t)$$

$$y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}$$

Closed-loop stability and transient response performance are governed by the $(n + 1)$ eigenvalues of the $(n + 1) \times (n + 1)$ closed-loop system

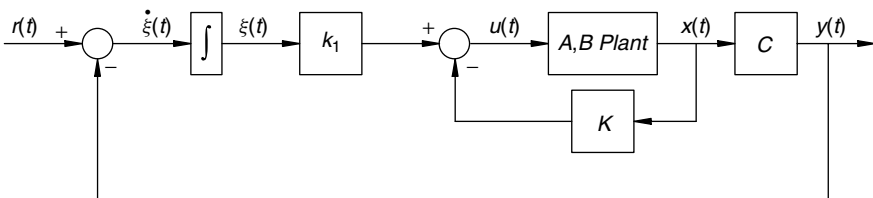


FIGURE 7.12 Closed-loop system block diagram.

dynamics matrix

$$\begin{bmatrix} A - BK & Bk_1 \\ -C & 0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} - \begin{bmatrix} B \\ 0 \end{bmatrix} [K \quad -k_1]$$

In order to arbitrarily place these closed-loop eigenvalues we require

$$\left(\begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix} \right)$$

to be a controllable pair, which we have not yet explicitly assumed. By assumption 2, the open-loop system dynamics matrix A is nonsingular, and hence $|A| \neq 0$ because a zero eigenvalue has been disallowed. Next, assumption 3 implies that

$$H(0) = C(sI - A)^{-1}B|_{s=0} = -CA^{-1}B \neq 0$$

For nonsingular A , the block determinant formula

$$\begin{aligned} \begin{vmatrix} A & B \\ -C & 0 \end{vmatrix} &= |CA^{-1}B| |A| \\ &= CA^{-1}B|A| \end{aligned}$$

shows that as a consequence of assumptions 2 and 3,

$$\begin{bmatrix} A & B \\ -C & 0 \end{bmatrix}$$

is a nonsingular $(n + 1) \times (n + 1)$ matrix. Analytical Exercise 7.1 asserts that controllability of the open-loop pair (A, B) as per assumption 1, along with nonsingularity of the preceding matrix, are necessary and sufficient for

$$\left(\begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix} \right)$$

to be a controllable pair. Consequently, under assumptions 1, 2, and 3, the $(n + 1)$ closed-loop eigenvalues can be placed arbitrarily by appropriate choice of the augmented feedback gain vector

$$[K \quad -k_1]$$

For instance, we can apply either the Bass-Gura formula or Ackermann's formula, as well as the MATLAB **place** function. Assuming from this point on that the gains have been chosen to yield an asymptotically stable closed-loop state equation, we must show next that for step reference input

$r(t) = R, t \geq 0$, we have

$$y_{ss} = \lim_{t \rightarrow \infty} y(t) = R$$

This involves checking that the closed-loop state equation has an equilibrium condition involving constant x_{ss}, ξ_{ss}, u_{ss} , along with $y_{ss} = R$. Using the relationship $u_{ss} = -Kx_{ss} + k_I \xi_{ss}$ leads to

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} A - BK & Bk_I \\ -C & 0 \end{bmatrix} \begin{bmatrix} x_{ss} \\ \xi_{ss} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} R \\ &= \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x_{ss} \\ \xi_{ss} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_{ss} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} R \\ &= \begin{bmatrix} A & B \\ -C & 0 \end{bmatrix} \begin{bmatrix} x_{ss} \\ u_{ss} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} R \end{aligned}$$

With A nonsingular and $H(0) = -CA^{-1}B \neq 0$, a block matrix inverse formula gives

$$\begin{bmatrix} A & B \\ -C & 0 \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} - A^{-1}B(CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(CA^{-1}B)^{-1} \\ (CA^{-1}B)^{-1}CA^{-1} & (CA^{-1}B)^{-1} \end{bmatrix}$$

which allows us to solve for $[x_{ss}^T, u_{ss}^T]^T$ to obtain

$$\begin{aligned} \begin{bmatrix} x_{ss} \\ u_{ss} \end{bmatrix} &= - \begin{bmatrix} A & B \\ -C & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ R \end{bmatrix} \\ &= \begin{bmatrix} A^{-1}B(CA^{-1}B)^{-1}R \\ -(CA^{-1}B)^{-1}R \end{bmatrix} \end{aligned}$$

in which, from before, $-(CA^{-1}B)^{-1} = H^{-1}(0)$, the inverse of the open-loop dc gain. We may now solve for ξ_{ss} , that is,

$$\begin{aligned} \xi_{ss} &= \frac{1}{k_I} [u_{ss} + Kx_{ss}] \\ &= \frac{1}{k_I} [1 - KA^{-1}B] H^{-1}(0)R \end{aligned}$$

We observe that $k_I \neq 0$, for otherwise we would have a closed-loop eigenvalue at $\lambda = 0$.

Finally,

$$\begin{aligned}
 y_{ss} &= Cx_{ss} \\
 &= C[A^{-1}B(CA^{-1}B)^{-1}R] \\
 &= (CA^{-1}B)(CA^{-1}B)^{-1}R \\
 &= R
 \end{aligned}$$

We therefore have achieved the desired steady-state tracking objective. All that is required is that the augmented feedback gain vector $[K \quad -k_I]$ stabilize the closed-loop state equation, thereby ensuring an asymptotically stable equilibrium for any constant reference input. This does not require perfect knowledge of the open-loop state equation. At this equilibrium, $\xi(t) = \xi_{ss}$ implies that $0 = \dot{\xi}(t) = R - y_{ss}$, so $y_{ss} = R$ is assured.

Example 7.9 We design a type I servomechanism for the state equation of Example 7.4 so that closed-loop unit step response reaches a steady-state value of 1 corresponding to zero steady-state error between the reference input $r(t)$ and the system output $y(t)$. We begin by checking that the three assumptions are satisfied. The state equation is in controller canonical form and is therefore controllable. We see from the associated transfer function given in Example 7.8 that there is neither a pole nor a zero at $s = 0$.

We may then proceed with the construction of a state feedback gain vector for the controllable pair

$$\begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -18 & -15 & -2 & 0 \\ \hline -1 & 0 & 0 & 0 \end{array} \right] \quad \begin{bmatrix} B \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

We select eigenvalues based on the ITAE criterion presented in Section 7.2 using an undamped natural frequency of $\omega_n = 2$ rad/s. This yields the desired fourth-order characteristic polynomial

$$\begin{aligned}
 s^4 + 2.1 \omega_n s^3 + 3.4 \omega_n^2 s^2 + 2.7 \omega_n^3 s + \omega_n^4 \\
 = s^4 + 4.2s^3 + 13.6s^2 + 21.6s + 16
 \end{aligned}$$

and associated eigenvalues

$$\lambda_{1,2} = -0.848 \pm j2.53 \quad \lambda_{3,4} = -1.25 \pm j0.828$$

We observe that, although we started with a state equation in controller canonical form, the four-dimensional controllable pair is not in controller

canonical form, so we must use our general purpose formulas to compute the state feedback gain vector. Either the Bass-Gura formula or Ackermann's formula yields

$$[K \quad -k_1] = [3.6 \quad -1.4 \quad 2.2 \mid -16]$$

and the associated control law is given by

$$\begin{aligned} \dot{\xi}(t) &= r(t) - y(t) \\ u(t) &= -[3.6 \quad -1.4 \quad 2.2] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + 16\xi(t) \end{aligned}$$

The resulting closed-loop state equation is

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{\xi}(t) \end{bmatrix} &= \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -21.6 & -13.6 & -4.2 & 16 \\ -1 & 0 & 0 & 0 \end{array} \right] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \xi(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r(t) \\ y(t) &= [1 \quad 0 \quad 0 \mid 0] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \xi(t) \end{bmatrix} \end{aligned}$$

To illustrate the robustness properties of type 1 servomechanisms, we consider a perturbation to the original system dynamics matrix that yields

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -16 & -16 & -1 \end{bmatrix}$$

having eigenvalues $-1, \pm j4$, indicating a marginally stable system. Using the same control law designed based on the nominal system dynamics matrix now yields the closed-loop system dynamics matrix

$$\begin{bmatrix} \tilde{A} - BK & Bk_1 \\ -C & 0 \end{bmatrix} = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -19.6 & -14.6 & -3.2 & 16 \\ -1 & 0 & 0 & 0 \end{array} \right]$$

with eigenvalues

$$\lambda_{1,2} = -0.782 \pm j3.15 \quad \lambda_{3,4} = -0.818 \pm j0.922$$

Thus, despite the perturbation, the closed-loop system remains asymptotically stable. The closed-loop unit step responses for both the nominal and perturbed cases are shown in Figure 7.13. We see that the perturbation has an effect on the transient response but, because closed-loop asymptotic stability has been maintained, zero steady-state error is still achieved as forced by the integral-error term in the control law.

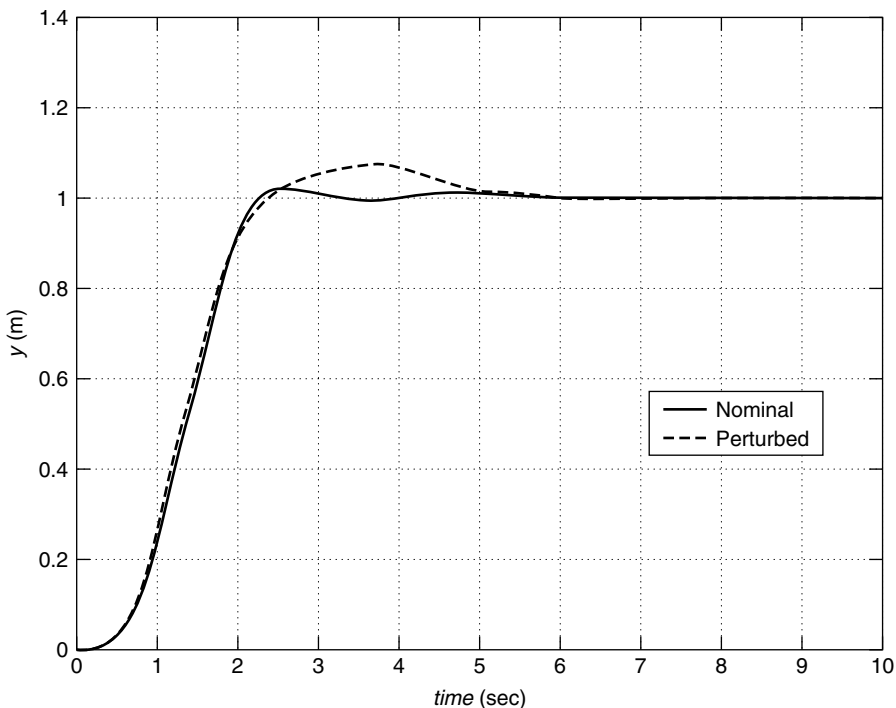


FIGURE 7.13 Nominal and perturbed closed-loop unit step responses for Example 7.9.

7.6 MATLAB FOR STATE FEEDBACK CONTROL LAW DESIGN

MATLAB for Shaping the Dynamic Response

The MATLAB functions that are useful for dynamic shaping are discussed in the MATLAB sections of Chapters 1 and 2. With either dynamic shaping

method (dominant/augmented eigenvalues or ITAE eigenvalues), a useful MATLAB capability is provided in conjunction with the step function:

```
figure;
DesSys = tf(numDes,denDes);
step(DesSys);
```

where **denDes** contains the $n + 1$ coefficients of the n th-order desired closed-loop characteristic polynomial (found based on dominant-plus-augmented eigenvalues or ITAE eigenvalues), and **numDes** is the constant desired behavior numerator, chosen to normalize the steady-state unit step response value to 1.0. After executing the step function with the desired system, one can right-click in the figure window, as described earlier, to display the performance measures on the plot (rise time, peak time, percent overshoot, and settling time). MATLAB calculates these values numerically from the response data by applying the definition for each performance measure; i.e., they should be accurate even for non-second-order systems.

MATLAB for Control Law Design and Evaluation

The following MATLAB functions are useful for design of linear state feedback control laws:

- K=place(A,B,DesEig)** Solve for state feedback gain matrix **K** to place the desired eigenvalues **DesEig** of the closed-loop system dynamics matrix $A - BK$.
- K=acker(A,B,DesEig)** Solve for state feedback gain matrix **K** to place the desired eigenvalues **DesEig** of the closed-loop system dynamics matrix $A - BK$ using Ackermann's formula (for single-input, single-output systems only).
- conv** Multiply polynomial factors to obtain a polynomial product.

Continuing MATLAB Example

Shaping the Dynamic Response For the Continuing MATLAB Example (rotational mechanical system), we compute two desired eigenvalues for control law design to improve the performance relative to the open-loop responses of Figure 2.2. We use a desired percent overshoot and settling time of 3 percent and 0.7 s, respectively, to find the

desired second-order control law eigenvalues. The following MATLAB code performs the dynamic shaping calculations for the Continuing MATLAB Example:

```
%-----
% Chapter 7. Dynamic Shaping
%-----

PO = 3;  ts = 0.7;           % Specify percent
                                % overshoot and settling
                                % time

term = pi^2 + log(PO/100)^2;
zeta = log(PO/100)/sqrt(term) % Damping ratio from PO
wn   = 4/(zeta*ts)           % Natural frequency from
                                % settling time and zeta

num2   = wn^2;               % Generic desired
                                % second-order system

den2   = [1 2*zeta*wn wn^2]
DesEig2 = roots(den2)        % Desired control law
                                % eigenvalues

Des2 = tf(num2,den2);        % Create desired system
                                % from num2 and den2

figure;
td = [0:0.01:1.5];
step(Des2,td);               % Right-click to get
                                % performance measures
```

This m-file generates the following results. It also generates the desired closed-loop response shown in Figure 7.14, with the performance specifications displayed via right-clicking. We see that the 3 percent overshoot is achieved exactly as in theory, whereas the settling time is close to the desired 0.7 s value.

```
zeta =
    0.7448

wn =
    7.6722

den2 =
    1.0000    11.4286    58.8627
```

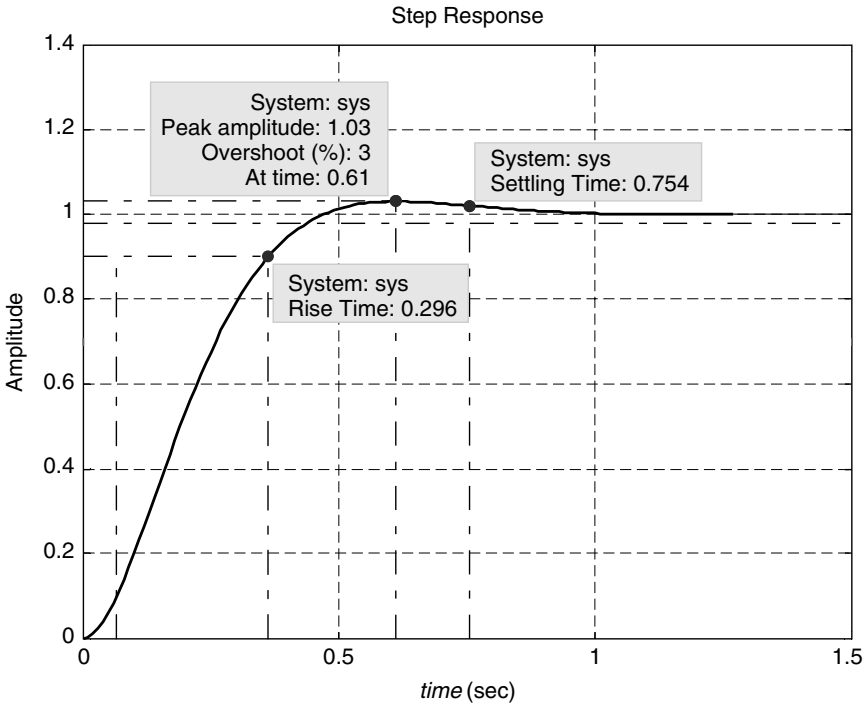


FIGURE 7.14 Desired second-order closed-loop response.

```
DesEig2 =
    -5.7143 + 5.1195i
    -5.7143 - 5.1195i
```

Control Law Design and Evaluation Next, we design a state feedback control law for the Continuing MATLAB Example. That is, we calculate the control law gain matrix K given A, B , and the desired control law eigenvalues developed in the previous subsection. The following MATLAB code segment performs this control law design for the continuing example:

```
%-----
% Chapter 7. Design of Linear State Feedback Control
% Laws
%-----
```



```

K      = place(A,B,DesEig2)           % Compute state
                                       % feedback gain matrix
                                       % K
Kack   = acker(A,B, DesEig2);        % Check K via
                                       % Ackermann's formula

Ac = A-B*K;   Bc = B;                % Compute closed-loop
                                       % state feedback system

Cc = C;       Dc = D;
JbkRc = ss(Ac,Bc,Cc,Dc);            % Create the
                                       % closed-loop
                                       % state-space system

[Yc,t,Xc] = lsim(JbkRc,U,t,X0);      % Compare open-loop and
                                       % closed-loop responses

figure;
subplot(211), plot(t,Xo(:,1),'r',t,Xc(:,1),'g'); grid;
axis([0 4 -0.2 0.5]);
set(gca,'FontSize',18);
legend('Open-loop','Closed-loop');
ylabel('\itx_1')
subplot(212), plot(t,Xo(:,2),'r',t,Xc(:,2),'g'); grid;
axis([0 4 -2 1]);
set(gca,'FontSize',18);
xlabel('\ittime (sec)'); ylabel('\itx_2');

```

This m-file, combined with the previous chapter m-files, yields the following output, plus the comparison of open- versus closed-loop state responses shown in Figure 7.15.

```

K =
18.86          7.43

```

Figure 7.15 shows that the simulated closed-loop system performs better than the open-loop system in terms of reduced overshoot and faster rise and settling times. There is less vibration with the closed-loop responses, and the steady-state zero values are obtained sooner than in the open-loop case.

Closed-loop system simulation also can be performed using MATLAB'S Simulink. Again, a detailed treatment of Simulink is beyond the scope of

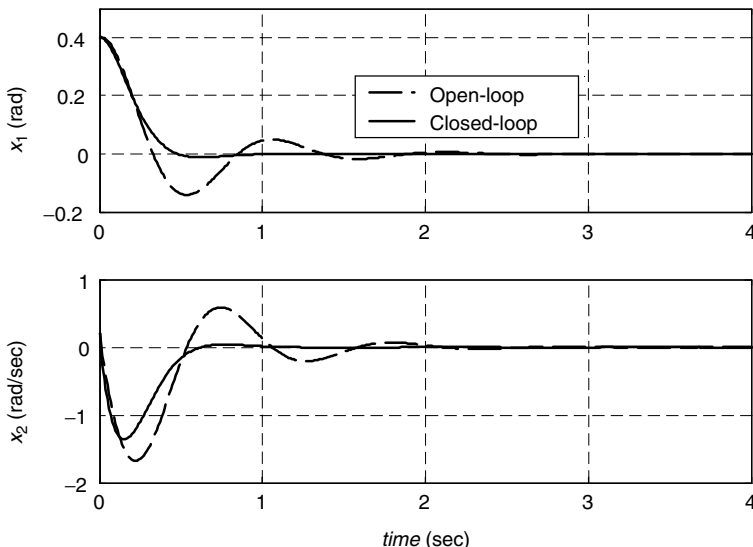


FIGURE 7.15 Open- versus closed-loop state responses for the Continuing MATLAB Example.

this book. The reader is encouraged to explore this powerful MATLAB toolbox on her or his own.

7.7 CONTINUING EXAMPLES: SHAPING DYNAMIC RESPONSE AND CONTROL LAW DESIGN

Continuing Example 1: Two-Mass Translational Mechanical System

Shaping the Dynamic Response For control law design, we need to specify desired eigenvalues that would improve the open-loop system performance. There are infinite possibilities; in this example we will use the two methods discussed in Section 7.2 to be applied later in control law design for Continuing Example 1.

We will use a dominant second-order system for desired eigenvalues to achieve a 5 percent overshoot and a 2 s settling time. These two eigenvalues must be augmented with two additional nondominant eigenvalues (we need 4 desired eigenvalues for our fourth-order system): real, negative, and at least 10 times farther to the left in the complex plane. We will use this specification of eigenvalues for multiple-input, multiple-output case *a*.

We also will use a fourth-order ITAE approach for desired eigenvalues, with the same undamped natural frequency ω_n as case *a* for easy

comparison. We will use this specification of eigenvalues for single-input, single-output case *b*.

Case a Percent overshoot is only a function of ξ ; substituting the desired 5 percent overshoot yields the dimensionless damping ratio $\xi = 0.69$. With this value, plus the desired 2 s settling time specification, we then find the undamped natural frequency $\omega_n = 2.90$ rad/s. This yields the desired dominant second-order transfer function, namely,

$$H(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{8.40}{s^2 + 4s + 8.40}$$

whose dominant, complex conjugate poles are $s_{1,2} = -2 \pm 2.10i$. The response of this desired dominant second-order system is shown in Figure 7.16 with the second-order performance measures. This figure was produced using the MATLAB step function with the preceding desired second-order numerator and denominator and then right-clicking to add the performance measures. As seen in Figure 7.16, we have obtained the

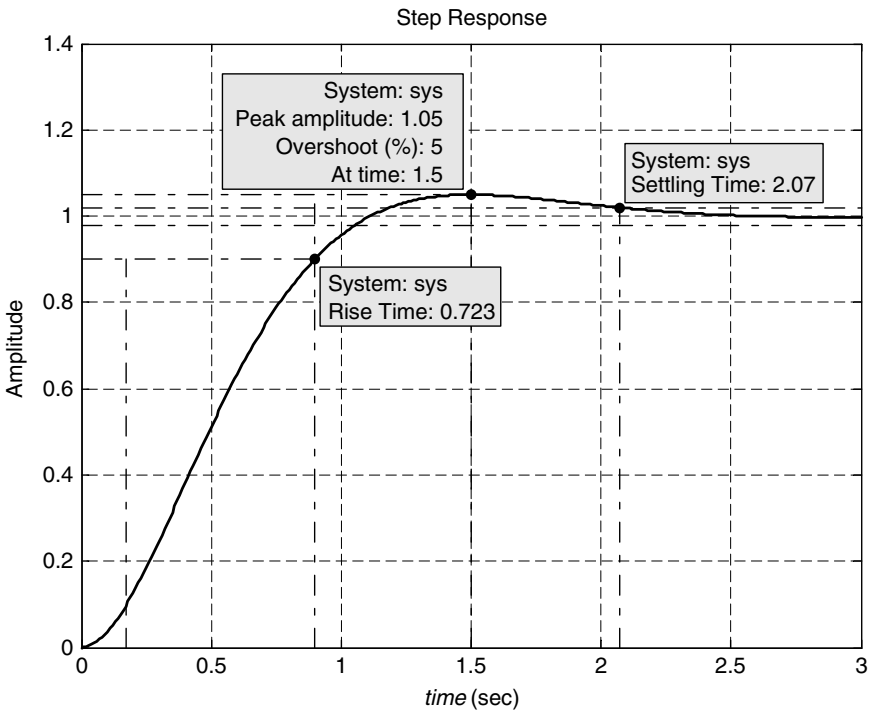


FIGURE 7.16 Dominant second-order response with performance characteristics.

desired 5 percent overshoot (at a peak time of 1.5 s) and a settling time of 2.07 s (2 s was specified). The 10 to 90 percent rise time is 0.723 s. We augment the dominant second-order eigenvalues to fit our fourth-order system as follows:

$$\lambda_{1,2} = -2 \pm 2.10i \quad \lambda_{3,4} = -20, -21$$

Note that we do not specify repeated eigenvalues for $\lambda_{3,4}$ to avoid any numerical issues with repeated roots. The transfer function for the fourth-order desired behavior that mimics second-order behavior is (normalized for a steady-state value of 1)

$$H_4(s) = \frac{3528}{s^4 + 45s^3 + 592s^2 + 2024s + 3528}$$

We will wait to plot the step response of this augmented fourth-order transfer function until the following subsection, where we compare all responses on one graph.

Case b. For a fourth-order system, the optimal ITAE characteristic polynomial is

$$\alpha_{\text{ITAE4}}(s) = s^4 + 2.1 \omega_n s^3 + 3.4 \omega_n^2 s^2 + 2.7 \omega_n^3 s + \omega_n^4$$

In this example we will use the same natural frequency from above, that is, $\omega_n = 2.90$ rad/s yielding

$$\alpha_{\text{ITAE4}}(s) = s^4 + 6.09s^3 + 28.56s^2 + 65.72s + 70.54$$

For this fourth-order desired characteristic polynomial, the four desired eigenvalues are

$$\lambda_{1,2} = -1.23 \pm 3.66i \quad \lambda_{3,4} = -1.81 \pm 1.20i$$

Figure 7.17 plots the fourth-order ITAE desired response, along with the dominant second-order and augmented fourth-order desired responses from case *a*. All are normalized to a steady-state value of 1.0 for easy comparison.

From Figure 7.17 we see that the augmented fourth-order response (solid) mimics the dominant second-order response (dashed) closely, as desired. The augmented fourth-order response lags the dominant second-order response, but it matches the required 5 percent overshoot and 2 s settling time well. The fourth-order ITAE response (dotted) did not

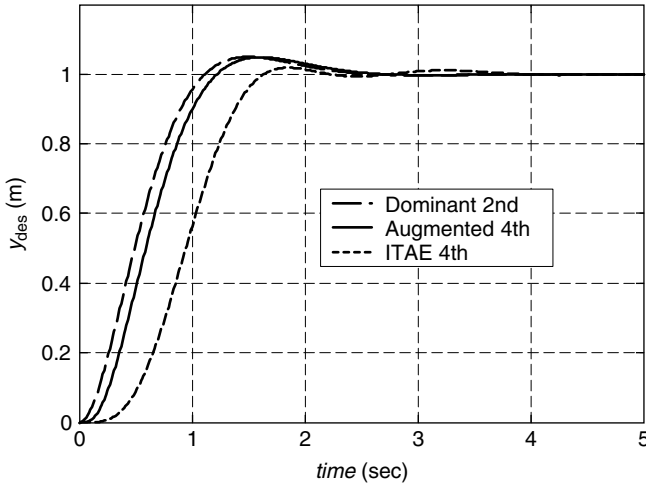


FIGURE 7.17 Dynamic shaping example results.

involve a percent overshoot or settling time specification, but we used the same natural frequency as in the dominant second-order response for comparison purposes. The ITAE response lags even further and demonstrates a fourth-order wiggle not present in the augmented fourth-order response.

Control Law Design and Evaluation For both cases of Continuing Example 1, we calculate a state feedback gain matrix K to move the closed-loop system eigenvalues as desired and hence achieve the control objectives. In both cases, we simulate the closed-loop behavior and compare the open-loop system responses with those of the closed-loop system.

For case *a* (multiple-input, multiple-output), we design the control law based on the desired eigenvalues developed earlier and we simulate the closed-loop system response given the same conditions as the open-loop simulation of Chapter 2: zero initial state and step inputs of magnitudes 20 and 10 N , respectively, for $u_1(t)$ and $u_2(t)$.

For case *b* [input $u_2(t)$ and output $y_1(t)$], we design the control law based on the desired eigenvalues developed earlier and we simulate the closed-loop system response given the same conditions as the open-loop simulation of Chapter 2: zero input $u_2(t)$ and initial state $x(0) = [0.1, 0, 0.2, 0]^T$.

Case a. The four desired eigenvalues corresponding to a desired 5% overshoot and 2 sec settling time were given earlier for case *a* : $\lambda_{1,2} =$

$-2 \pm 2.10i$ and $\lambda_{3,4} = -20, -21$. By using the MATLAB function **place** we find the following 2×4 state feedback gain matrix K :

$$K = \begin{bmatrix} 606 & 858 & 637 & 47 \\ -3616 & -166 & 759 & 446 \end{bmatrix}$$

On checking, the eigenvalues of $A - BK$ are indeed those specified in the **place** command. The reference input $r(t)$ we take to be the same as the open-loop input $u(t)$, i.e., step inputs of magnitudes 20 and 10 N, respectively, for $r_1(t)$ and $r_2(t)$. Simulating the closed-loop system response and comparing it with the open-loop system response yields Figure 7.18.

The first thing evident in Figure 7.18 is that the closed-loop control law attenuates the output (see Section 7.4). This is so because the control law in effect adds “virtual” springs in addition to the “real” springs in the system model; a stiffer set of springs will cause the output attenuation seen in Figure 7.18. Before we can discuss the performance of the control law, we must ensure that the steady-state closed-loop response match that of the open-loop system. This can be done in two ways: First, the output attenuation correction factors (simple term-by-term input gains) are 14.3 and 3.99 for outputs $y_1(t)$ and $y_2(t)$, respectively. The

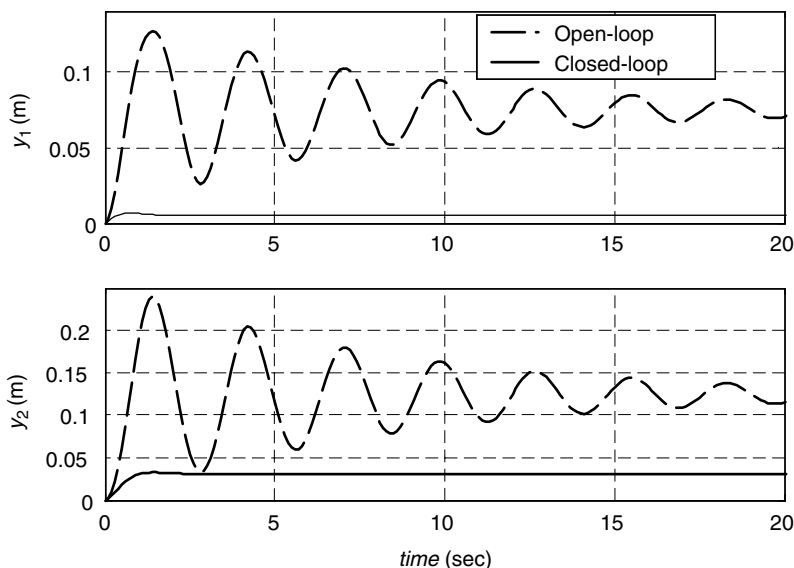


FIGURE 7.18 Open- versus closed-loop responses for case a, with output attenuation.

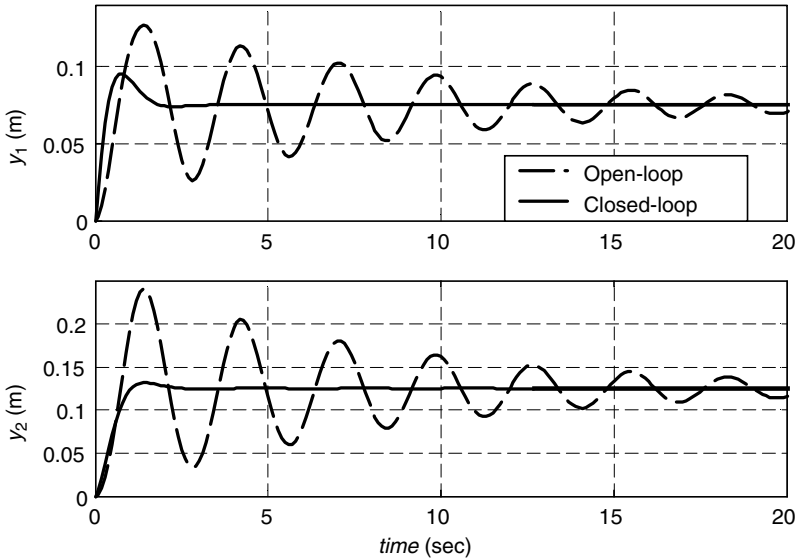


FIGURE 7.19 Open- versus closed-loop responses for case a corrected via scalar input gains.

corrected closed-loop responses following this approach are shown in Figure 7.19. Second, using the modified control law with attenuation matrix G and reference input equal to the open-loop steady state values ($R = [0.075, 0.125]^T$), results in the corrected closed-loop responses of Figure 7.20. The input gain matrix G is

$$G = \begin{bmatrix} 1206 & 437 \\ -3816 & 959 \end{bmatrix}$$

Now we see in both Figure 7.19 and Figure 7.20 that the designed control law has improved the performance of the open-loop system significantly. The settling time of 2 s has been achieved, so the closed-loop system responses approach their steady-state values much sooner than those of the open-loop system. However, in the top plot of Figure 7.19 we see that the percent overshoot is much greater than the specified 5 percent. This was not visible in the top plot of Figure 7.18 because of the vertical axis scaling. This is a well-known problem from classical controls—there are zeros (numerator roots) present in this system. The dominant second-order eigenvalue specification method does not account for any zeros; thus the results are skewed in the presence of zeros. In classical controls, a way to handle this is preshaping the input via filters.

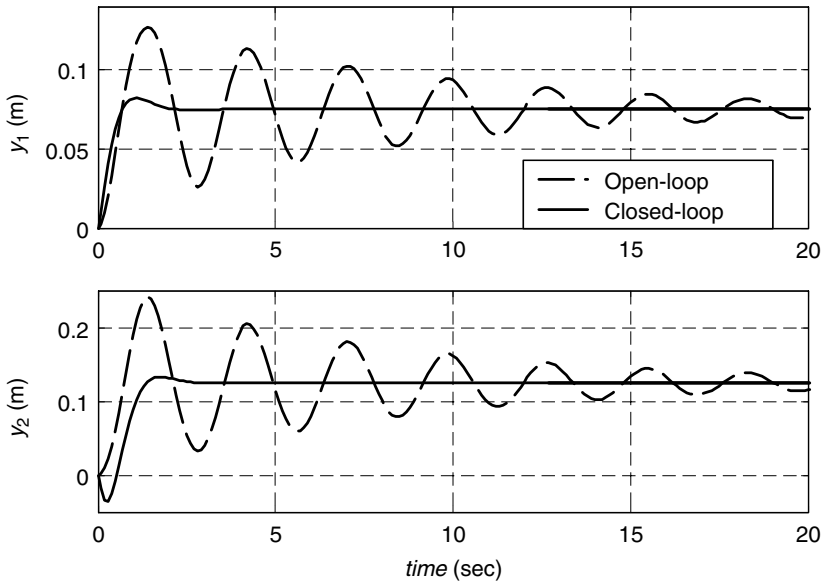


FIGURE 7.20 Open- versus closed-loop responses for case *a* corrected via input gain matrix.

Note that this overshoot problem is less severe when using the more general G -method as shown in Figure 7.20. However, for the output $y_2(t)$, there is a noticeable undershoot before it attains the desired value of 0.125.

Case *b*. The four desired closed-loop eigenvalues derived from the fourth-order ITAE characteristic polynomial are $\lambda_{1,2} = -1.23 \pm 3.66i$ and $\lambda_{3,4} = -1.81 \pm 1.20i$. Again using the MATLAB function `place`, we find the following 1×4 state feedback gain vector K :

$$K = [-145 \quad -61 \quad 9 \quad 97]$$

Since case *b* is a single-input, single-output situation, we can check this result using Ackermann's formula (MATLAB function `acker`); the results are identical. A quick check confirms that the eigenvalues of $A - BK$ are indeed those specified at the outset. For zero reference input and initial state $x(0) = [0.1, 0, 0.2, 0]^T$, the closed-loop state variable responses are compared with the open-loop state variable responses in Figure 7.21.

Since the closed-loop state variables all tend to zero in steady state as the open-loop states do, there is no output attenuation issue. We see in

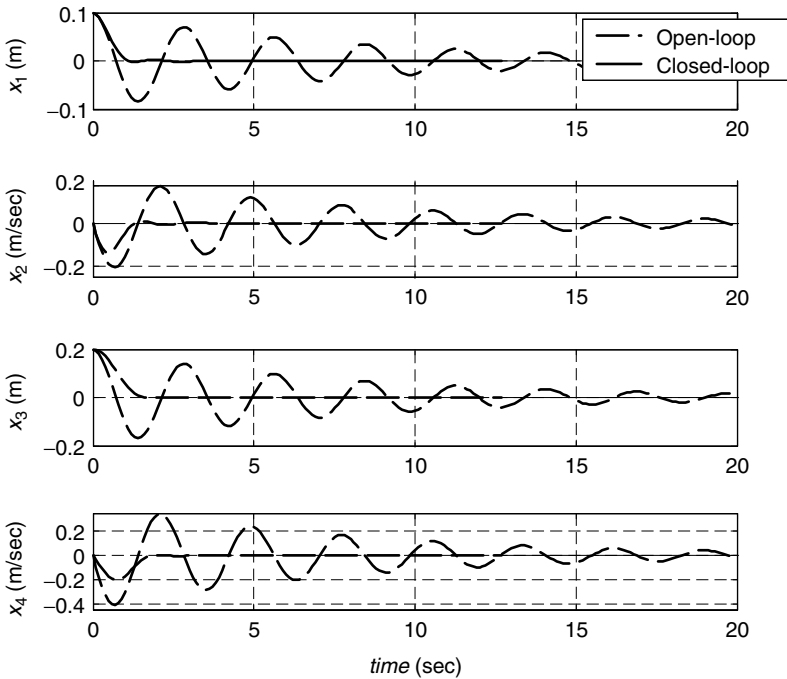


FIGURE 7.21 Open- versus closed-loop responses for case *b*.

Figure 7.21 that the designed control law has improved the performance of the open-loop system significantly. The closed-loop system responses meet their zero steady-state values much sooner than those of the open-loop system. The displacements do not overshoot significantly, and the velocity responses are both better than their open-loop counterparts.

Continuing Example 2: Rotational Electromechanical System

Shaping the Dynamic Response For control law design, we must specify desired eigenvalues to improve the open-loop system performance. In this example we will use a desired dominant first-order system, to be applied later in control law design for Continuing Example 2. We will use a dominant first-order system with time constant $\tau = \frac{1}{4}$ sec. We then augment the dominant eigenvalue associated with this time constant with two additional nondominant eigenvalues (we need three desired eigenvalues for our third-order system): real, negative, and larger amplitude so that their effect does not change the dominant behavior very much.

The relationship between the desired dominant eigenvalue a and first-order time constant τ is $e^{at} = e^{-t/\tau}$; therefore, $a = -1/\tau = -4$. This

yields the desired dominant first-order transfer function:

$$H(s) = \frac{-a}{s - a} = \frac{4}{s + 4}$$

Let us augment the dominant first-order eigenvalue to fit our third-order system desired eigenvalues requirement as follows: We choose additional eigenvalues approximately three times farther to the left to yield

$$\lambda_{1,2,3} = -4, -12, -13$$

Note that we do not specify repeated eigenvalues for $\lambda_{2,3}$ to avoid any numerical issues with repeated roots. The transfer function for the third-order desired behavior that mimics first-order behavior is (normalizing for a steady-state value of 1):

$$H_3(s) = \frac{624}{s^3 + 29s^2 + 256s + 624}$$

Figure 7.22 plots the augmented third-order desired response, along with the dominant first-order system from which it was derived. Both are normalized to a steady-state value of 1.0 for easy comparison. From Figure 7.22 we see that the augmented third-order response (solid) mimics the dominant first-order response (dashed) fairly closely. We see in the dashed curve that after three time constants (at $t = 0.75$ s), the dominant first-order response has achieved 95 percent of the steady-state value of 1.0. The augmented third-order response lags the dominant first-order

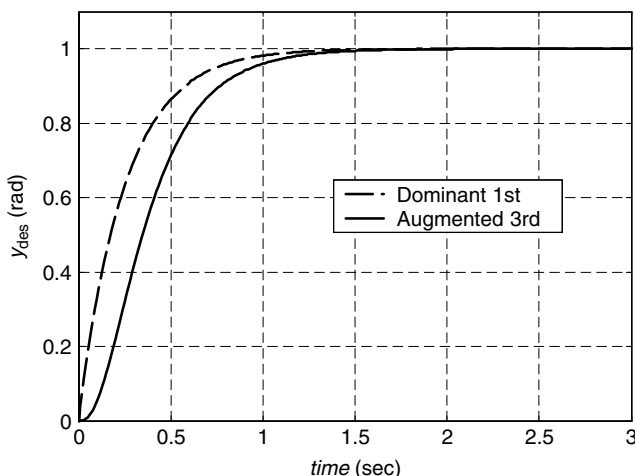


FIGURE 7.22 Dynamic shaping for Continuing Example 2.

response; we can make this arbitrarily close to the dashed curve by moving the augmented eigenvalues even farther to the left. However, this may lead to large numerical values for control gains, which is generally to be avoided. This could potentially require large actuator amplitudes, perhaps exceeding physical limits.

Control Law Design and Evaluation

For Continuing Example 2, we compute a state feedback gain matrix K to move the closed-loop system eigenvalues where desired and hence achieve the control objectives. Then we simulate the closed-loop behavior and compare the open-loop system responses with those of the closed-loop system.

We design the control law based on the desired eigenvalues developed in the preceding subsection, and we simulate the closed-loop system response given the same conditions as the open-loop simulation of Chapter 2 [zero initial state and a unit step input in voltage $v(t)$].

The three desired eigenvalues for this case, $\lambda_{1,2,3} = -4, -12, -13$, were specified by a dominant first-order eigenvalue augmented by two more real, negative eigenvalues farther to the left by a factor of three. By using MATLAB functions **place** or **acker** we find the following 1×3 state feedback gain matrix K :

$$K = [312 \quad 127 \quad 13]$$

The eigenvalues of $A - BK$ are indeed those that were specified, as the reader may verify. Simulating the closed-loop system response for zero initial state and unit step reference input and comparing with the open-loop unit step response yields Figure 7.23.

There is no output attenuation issue because the open-loop response increases linearly after the transient response has died out; this is as expected because there is no torsional spring in the system model. However, we could use the input gain matrix method to achieve any desired steady-state angular displacement for the closed-loop system.

The closed-loop angular displacement $x_1(t)$ in Figure 7.23 (top plot) was artificially scaled to achieve a steady-state value of 0.5 rad. Comparing the open- and closed-loop angular displacement, we see that the state feedback control law has effectively added a virtual spring whereby we can servo to commanded angles rather than having the shaft angle increase linearly without bound as in the open-loop case. In Figure 7.23, the closed-loop angular velocity and acceleration both experience a transient and then tend to zero steady-state values. The open-loop values are the same as those plotted in Figure 2.5.

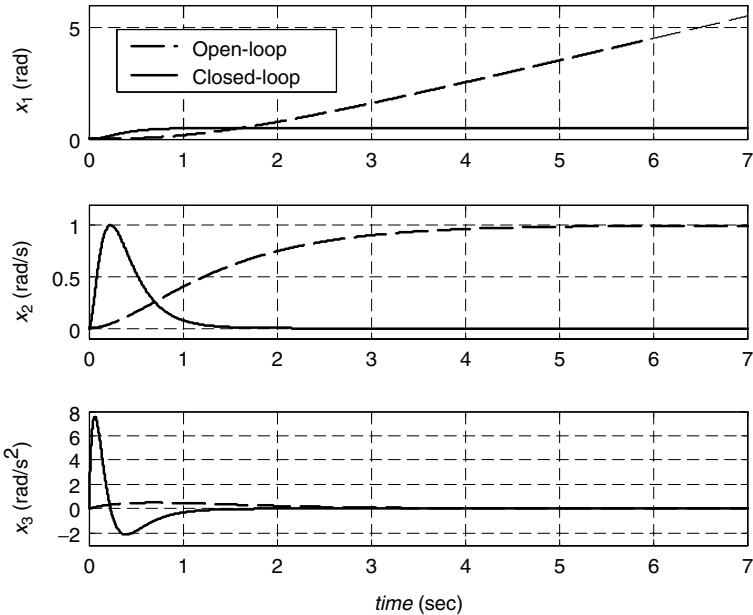


FIGURE 7.23 Open- versus closed-loop responses for Continuing Example 2.

For this example, the closed-loop system dynamics matrix is

$$A - BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -624 & -256 & -29 \end{bmatrix}$$

Now the (3,1) element of $A - BK$ is no longer 0 as it was for the open-loop A matrix; this nonzero term represents the virtual spring provided by the control law, allowing constant commanded shaft angles to be achieved.

The coefficients of the closed-loop characteristic polynomial can be seen in the third row of $A - BK$, in ascending order of powers of s , with negative signs. The closed-loop system is asymptotically stable, changed from the marginally stable open-loop system because all three eigenvalues are now negative real numbers.

7.8 HOMEWORK EXERCISES

Numerical Exercises

NE7.1 For the following cases, determine acceptable closed-loop system eigenvalues to achieve the required behavior. In each case, plot

the unit step response to demonstrate that the desired behavior is approximately satisfied.

- Determine acceptable eigenvalues for a second-, third-, and fourth-order system to approximate a first-order system with a time constant of 0.5 s.
- Determine acceptable eigenvalues for a second-, third-, and fourth-order system to approximate a second-order system with a percent overshoot of 6 percent and a settling time of 4 s.
- Co-plot the desired ITAE responses for second-, third-, and fourth-order systems assuming $\omega_n = 5$ rad/s. Discuss your results.

NE7.2 For each (A, B) pair below, use the Bass-Gura formula to calculate the state feedback gain vector K to place the given eigenvalues of the closed-loop system dynamics matrix $A - BK$. Check your results.

$$\text{a. } A = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \lambda_{1,2} = -2 \pm 3i$$

$$\text{b. } A = \begin{bmatrix} 0 & 1 \\ -6 & -8 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \lambda_{1,2} = -4, -5$$

$$\text{c. } A = \begin{bmatrix} 0 & 1 \\ -6 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \lambda_{1,2} = -4, -5$$

$$\text{d. } A = \begin{bmatrix} 0 & 8 \\ 1 & 10 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \lambda_{1,2} = -1 \pm i$$

NE7.3 Repeat **NE 7.2** using Ackermann's Formula.

Analytical Exercises

AE7.1 Show that the $(n + 1)$ -dimensional single-input, single-output state equation

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t)$$

with output $\xi(t)$ is controllable if and only if the n -dimensional single-input, single-output state equation represented by (A, B, C) is controllable and the $(n + 1) \times (n + 1)$ matrix

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$$

is nonsingular. What is the interpretation of the state variable $\xi(t)$?

- AE7.2** Suppose that the pair (A, B) is controllable, and for any finite $t_f > 0$, consider the state feedback gain matrix $K = -B^T W^{-1}(0, t_f)$. Show that the closed-loop state equation

$$\dot{x}(t) = (A - BK)x(t)$$

is asymptotically stable. Hint: Recall AE6.5.

- AE7.3** Suppose that given the symmetric positive definite matrix Q , the Riccati matrix equation

$$A^T P + PA - PBB^T P + Q = 0$$

has a symmetric positive-definite solution P . Show that the state feedback law $u(t) = -K x(t)$ with $K = -B^T P$ yields the closed-loop state equation

$$\dot{x}(t) = (A - BK)x(t)$$

that is asymptotically stable, thereby implying that the pair (A, B) is stabilizable.

- AE7.4** For the linear state equation (7.1) and state feedback law (7.4), show that the open- and closed-loop transfer functions are related by

$$H_{CL}(s) = H(s)[I + K(sI - A)^{-1}B]^{-1}G$$

This indicates that the same closed-loop input-output behavior can be achieved using a dynamic precompensator instead of state feedback. Hint: Use AE1.8.

Continuing MATLAB Exercises

- CME7.1** For the system given in CME1.1:

- Determine the desired eigenvalues for a generic second-order system to obtain 1 percent overshoot and a 1-s settling time. Plot the unit step response for this desired behavior and discuss your results.
- Design a state feedback control law to achieve the eigenvalue placement of part a. Compare open- and closed-loop responses to a unit step input, assuming zero initial conditions. Normalize the closed-loop output level to match that of

the open-loop system for easy comparison. Were your control law design goals met?

- c. Design a state feedback servomechanism to achieve the eigenvalue placement of part a and a steady-state output value of 1. Compare open- and closed-loop responses to a unit step input. In this case, normalize the open-loop output level to 1 for easy comparison. Were your control law design goals met?

CME7.2 For the system given in CME1.2:

- a. Determine the desired eigenvalues for a first-order system with a time constant of 1 s, augmented by two additional real, negative eigenvalues (the first exactly 10 times the dominant eigenvalue, the third one less than the second). Plot the unit step response for this desired behavior, and discuss your results.
- b. Design a state feedback control law to achieve the eigenvalue placement of part a. Compare open- and closed-loop responses to a unit step input. Normalize the closed-loop output level to match that of the open-loop system for easy comparison. Were your control law design goals met?
- c. Design a state feedback servomechanism to achieve the eigenvalue placement of part a and a steady-state output value of 1. Compare open- and closed-loop responses to a unit step input. In this case, normalize the open-loop output level to 1 for easy comparison. Were your control law design goals met?

CME7.3 For the system given in CME1.3:

- a. Determine the desired eigenvalues for a fourth-order ITAE system with $\omega_n = 2$ rad/s. Plot the unit step response for this desired behavior, and discuss your results.
- b. Design a state feedback control law to achieve the eigenvalue placement of part a. Compare open- and closed-loop responses to a unit step input. Normalize the closed-loop output level to match that of the open-loop system for easy comparison. Were your control law design goals met?
- c. Design a state feedback servomechanism to achieve the eigenvalue placement of part a and a steady-state output value of 1. Compare open- and closed-loop responses to a unit step input. In this case normalize, the open-loop output

level to 1 for easy comparison. Were your control law design goals met?

CME7.4 For the system given in CME1.4:

- a) Determine the desired eigenvalues for a generic second-order system to obtain 2 percent overshoot and a 2-s settling time. Plot the unit step response for this desired behavior and discuss your results.
- b) Design a state feedback control law to achieve the eigenvalue placement of part a. Compare open- and closed-loop responses to a unit step input. Normalize the closed-loop output level to match that of the open-loop system for easy comparison. Were your control law design goals met?
- c) Design a state feedback servomechanism to achieve the eigenvalue placement of part a and a steady-state output value of 1. Compare open- and closed-loop responses to a unit step input. In this case, normalize the open-loop output level to 1 for easy comparison. Were your control law design goals met?

Continuing Exercises

CE7.1a Given control law design criteria of 3 percent overshoot and a 3-s settling time: (i) Calculate the undamped natural frequency and damping ratio required for a standard generic second-order system to achieve this (assuming a unit step input). What are the associated system eigenvalues? Plot the unit step response for the result and demonstrate how well the design criteria are met (normalize your output to ensure the final value is 1.0). Display the resulting rise time, peak time, settling time, and percent overshoot on your graph. (ii) Augment these desired second-order system eigenvalues for control law design in CE1; since this is a sixth-order system you, will need four additional eigenvalues (real, negative, approximately 10 times farther to the left). Choose the first additional eigenvalue to be exactly ten times the real part of the dominant second-order system eigenvalues. For the remaining three, successively subtract one from the first additional eigenvalue (to avoid repeated eigenvalues). (iii) Also compute the optimal ITAE sixth-order coefficients and eigenvalues using an undamped natural frequency twice that from the dominant second-order approach. Plot both the sixth-order ITAE

and the augmented sixth-order desired step responses with the dominant second-order step response of part (i) (normalize to ensure steady-state values of 1.0); compare and discuss.

- CE7.1b** For the desired closed-loop eigenvalues from CE7.1a, design state feedback control laws (i.e., calculate K) for all three cases from CE2.1. For cases (i) and (ii), use the augmented sixth-order eigenvalues based on dominant second-order behavior; for case (iii), use the sixth-order ITAE eigenvalues. In each case, evaluate your results: Plot and compare the simulated open- versus closed-loop output responses for the same input cases as in CE2.1a; use output attenuation correction so that the closed-loop steady-state values match the open-loop steady-state values for easy comparisons.
- CE7.2a** Since this is a fourth-order system, we will need four desired eigenvalues for future control law design in CE2. Use a fourth-order ITAE approach with undamped natural frequency $\omega_n = 3$ rad/s to generate the four desired eigenvalues. Plot the desired system impulse response.
- CE7.2b** For the desired closed-loop eigenvalues from CE7.2a, design state feedback control laws (i.e., calculate K) for all three cases from CE2.2. In each case, evaluate your results: Plot and compare the simulated open- versus closed-loop output responses for the same input cases as in CE2.2a. Be sure to scale the vertical axes so that the closed-loop responses are clearly visible.
- CE7.3a** Use a dominant first-order system with time constant $\tau = 0.5$ s. What is the associated desired eigenvalue? Augment this desired first-order eigenvalue for future control law design in CE3; since this is a third-order system, you will need two additional eigenvalues (real, negative, approximately 10 times farther to the left). Choose the first additional eigenvalue to be exactly 10 times the dominant first-order eigenvalue. For the remaining eigenvalue, subtract one from the first additional eigenvalue (to avoid repeated eigenvalues). Plot this augmented third-order desired step response versus the dominant first-order step response (normalize to ensure steady-state values of 1.0); compare and discuss.
- CE7.3b** For the desired closed-loop eigenvalues from CE7.3a, design state feedback control laws (i.e., calculate K) for both cases from CE2.3. In each case, evaluate your results: Plot and compare the simulated open- versus closed-loop output responses for the same

input cases as in CE2.3a [for case (ii), use output attenuation correction so that the closed-loop steady-state values match the open-loop steady-state values for easy comparison].

- CE7.4a** Based on your knowledge of the CE4 system and the methods of this chapter, calculate a good set of desired closed-loop eigenvalues to be used as input to the control law design problem. Plot the responses using appropriate input signals, and discuss your results.
- CE7.4b** For your desired closed-loop eigenvalues from CE7.4a, design a state feedback control law (i.e., calculate K). Evaluate your results: Plot and compare the simulated open- versus closed-loop output responses for a unit impulse input and zero initial conditions as in CE2.4a.
- CE7.5** In Chapter 5 we found that the CE1.5 system was not minimal. However, since the open-loop system is controllable, for this problem, use the fourth-order system for control law design, i.e., do not use the minimal second-order system. In this way you can preserve the physical interpretation of the four state variables.
- CE7.5a** Based on your knowledge of the CE5 system and the methods of this chapter, compute a good set of desired closed-loop eigenvalues to be used as input to the control law design problem. Plot the responses using appropriate input signals, and discuss your results.
- CE7.5b** For your desired closed-loop eigenvalues from CE7.5a, design a state feedback control law (i.e., calculate K). Evaluate your results: Plot and compare the simulated open- versus closed-loop output responses for the initial conditions given in CE2.5a.