3. Establish the base coordinate system such that the z_0-axis is aligned with the first joint axis, the x_0-axis is perpendicular to the z_0-axis, and the y_0-axis is determined by the right-hand rule.
4. Establish the nth hand coordinate system such that the x_n-axis is perpendicular to the last joint axis. The z_n-axis is usually chosen in the direction of approach of the end effector.
5. Attach a Cartesian coordinate system to the distal end of all the other links as follows:
   - The z_i-axis is aligned with the $(i + 1)$th joint axis.
   - The x_i-axis is defined along the common normal between the ith and $(i + 1)$th joint axes, pointing from the ith to the $(i + 1)$th joint axis. If the joint axes are parallel, the x_i-axis can be chosen anywhere perpendicular to the two joint axes. In the case of two intersecting joint axes, the x_i-axis can be defined either in the direction of the vector cross product $\mathbf{z}_{i-1} \times \mathbf{z}_i$ or in the opposite direction, and the origin is located at the point of intersection.
   - The y_i-axis is defined according to the right-hand rule.
6. Determine the link parameters and joint variables, a_i, α_i, θ_i, and d_i.

There are $n + 1$ coordinate systems for an n-dof manipulator. However, if additional reference coordinate systems are defined, they can be related to one of the coordinate systems above by a transformation matrix. We note that John Craig used a different convention; he attached the ith coordinate system to the proximal end of link i, which results in a different homogeneous transformation matrix.

2.3 DENAVIT–HARTENBERG HOMOGENEOUS TRANSFORMATION MATRICES

Having established a coordinate system to each link of a manipulator, a 4 × 4 transformation matrix relating two successive coordinate systems can be established. Observation of Fig. 2.2 reveals that the ith coordinate system can be thought of as being displaced from the $(i - 1)$th coordinate system by the following successive rotations and translations.

1. The $(i - 1)$th coordinate system is translated along the z_{i-1}-axis a distance d_i. This brings the origin O_{i-1} into coincidence with H_{i-1}. The corresponding transformation matrix is

$T(z,$

2. The displaced $(i - 1)$th c
an angle θ_i, which brings
The corresponding transf

$T(z, \theta$

3. The displaced $(i - 1)$th c
a distance a_i. This brings
corresponding transform

$T(x$

4. The displaced $(i - 1)$th
an angle α_i, which brin
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We may think of the transf
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matrix, ${}^{i-1}A_i$, is given by

${}^{i-1}A_i = T($

Expanding Eq. (2.1), we obtai

${}^{i-1}A_i = \begin{bmatrix} c\theta_i \\ s\theta_i \\ 0 \\ 0 \end{bmatrix}$

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pendicular to the z_0-axis, and
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ı such that the x_n-axis is per-
axis is usually chosen in the

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defined either in the direction
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coincidence with H_{i-1}. The

$$T(z,d) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

2. The displaced $(i-1)$th coordinate system is rotated about the z_{i-1}-axis an angle θ_i, which brings the x_{i-1}-axis into alignment with the x_i-axis. The corresponding transformation matrix is

$$T(z,\theta) = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3. The displaced $(i-1)$th coordinate system is translated along the x_i-axis a distance a_i. This brings the origin O_{i-1} into coincidence with O_i. The corresponding transformation matrix is

$$T(x,a) = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

4. The displaced $(i-1)$th coordinate system is rotated about the x_i-axis an angle α_i, which brings the two coordinate systems into complete coincident. The corresponding transformation matrix is

$$T(x,\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\alpha_i & -s\alpha_i & 0 \\ 0 & s\alpha_i & c\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We may think of the transformations above as four basic transformations about the moving coordinate axes. Therefore, the resulting transformation matrix, ${}^{i-1}A_i$, is given by

$$ {}^{i-1}A_i = T(z,d)T(z,\theta)T(x,a)T(x,\alpha). \qquad (2.1)$$

Expanding Eq. (2.1), we obtain

$$ {}^{i-1}A_i = \begin{bmatrix} c\theta_i & -c\alpha_i s\theta_i & s\alpha_i s\theta_i & a_i c\theta_i \\ s\theta_i & c\alpha_i c\theta_i & -s\alpha_i c\theta_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (2.2)$$

Equation (2.2) is called the *Denavit–Hartenberg (D-H) transformation matrix*. The trailing subscript i and the leading superscript $i-1$ denote that the transformation takes place from the ith coordinate system to the $(i-1)$th coordinate system.

Let the homogeneous coordinates of the position vector of a point relative to the ith coordinate system be denoted by ${}^{i}\mathbf{p} = [p_x, p_y, p_z, 1]^{\mathrm{T}}$. Also let the homogeneous coordinates of a unit vector expressed in the ith coordinate system be denoted by ${}^{i}\mathbf{u} = [u_x, u_y, u_z, 0]^{\mathrm{T}}$. Then the transformation of a position vector and a unit vector from the ith to the $(i-1)$th coordinate system can be written as

$$ {}^{i-1}\mathbf{p} = {}^{i-1}A_i\,{}^{i}\mathbf{p}, \tag{2.3} $$

$$ {}^{i-1}\mathbf{u} = {}^{i-1}A_i\,{}^{i}\mathbf{u}. \tag{2.4} $$

Note that the leading superscript is used to indicate the coordinate system with respect to which a vector is expressed. Although the transformation matrix A is not orthogonal, the inverse transformation exists and is given by

$$ {}^{i}A_{i-1} = ({}^{i-1}A_i)^{-1} = \begin{bmatrix} c\theta_i & s\theta_i & 0 & -a_i \\ -c\alpha_i s\theta_i & c\alpha_i c\theta_i & s\alpha_i & -d_i s\alpha_i \\ s\alpha_i s\theta_i & -s\alpha_i c\theta_i & c\alpha_i & -d_i c\alpha_i \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.5} $$

Example 2.3.1 *Planar 3-DOF Manipulator* Figure 2.3 shows a 3-dof planar manipulator constructed with three revolute joints located at points O_0, A, and P, respectively. A coordinate system is attached to each link. The (x_0, y_0, z_0) coordinate system is attached to the base with its origin located at the first joint pivot and the x-axis pointing to the right. Since the joint axes are all parallel to each other, all the twist angles α_i and translational distances d_i are zero.

For the coordinate systems chosen, the link parameters are given in Table 2.1. The D-H transformation matrices are obtained by substituting the D-H link parameters into Eq. (2.2):

$$ {}^{0}A_1 = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & a_1 c\theta_1 \\ s\theta_1 & c\theta_1 & 0 & a_1 s\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.6} $$

berg (D-H) *transformation ma-*
uperscript $i-1$ denote that the
rdinate system to the $(i-1)$th

osition vector of a point relative
${}^i\mathbf{p} = [p_x, p_y, p_z, 1]^T$. Also let
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th to the $(i-1)$th coordinate

, (2.3)

. (2.4)

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lthough the transformation ma-
ation exists and is given by

$$\begin{matrix} 0 & -a_i \\ s\alpha_i & -d_i s\alpha_i \\ c\alpha_i & -d_i c\alpha_i \\ 0 & 1 \end{matrix}\Bigg] . \quad (2.5)$$

Figure 2.3 shows a 3-dof
volute joints located at points
m is attached to each link. The
base with its origin located at
the right. Since the joint axes
α_i and translational distances

ink parameters are given in
are obtained by substituting

$$\begin{matrix} a_1 c\theta_1 \\ a_1 s\theta_1 \\ 0 \\ 1 \end{matrix}\Bigg] , \quad (2.6)$$

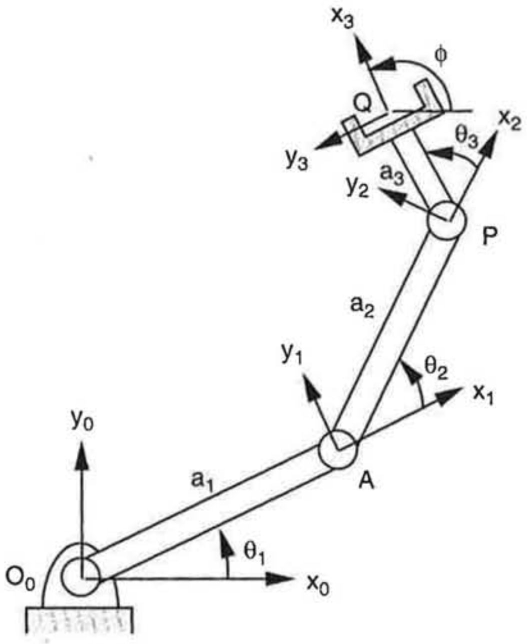

FIGURE 2.3. Planar 3-dof manipulator.

$${}^1A_2 = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & a_2c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & a_2s\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (2.7)$$

$${}^2A_3 = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_3c\theta_3 \\ s\theta_3 & c\theta_3 & 0 & a_3s\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (2.8)$$

Example 2.3.2 *SCARA Arm* The SCARA arm is an important type of 4-dof manipulator. It has been produced by several companies, including Adept

TABLE 2.1. D-H Parameters of a 3-DOF Manipulator

Joint i	α_i	a_i	d_i	θ_i
1	0	a_1	0	θ_1
2	0	a_2	0	θ_2
3	0	a_3	0	θ_3

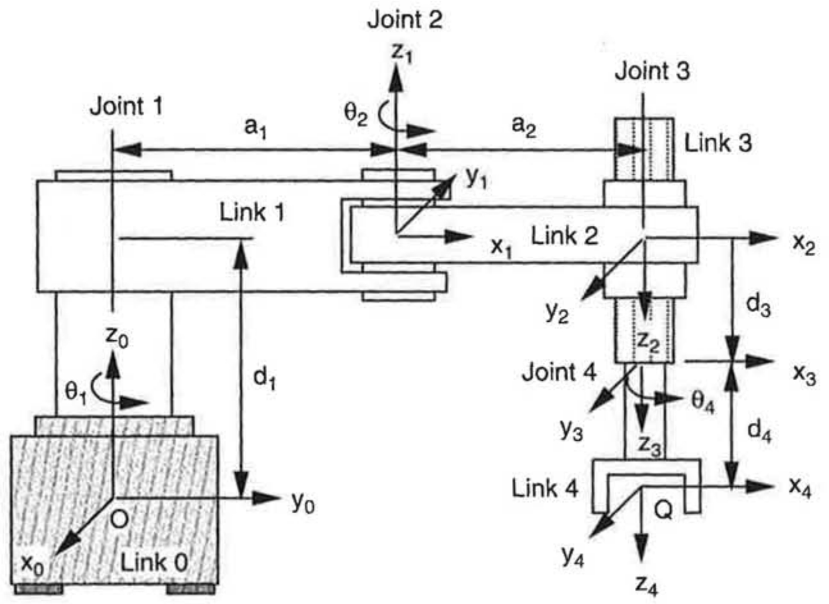

FIGURE 2.4. Schematic diagram of a SCARA arm.

Technology, IBM, Seiko, and others. A SCARA arm is constructed with four joint axes parallel to each other. The first two and the fourth are revolute joints, and the third is a prismatic joint. Figure 2.4 shows a schematic diagram of a SCARA arm. For the coordinate systems established in the figure, the corresponding link parameters are listed in Table 2.2.

Substituting the D-H link parameters into Eq. (2.2), we obtain the D-H transformation matrices:

$$
{}^0A_1 = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & a_1 c\theta_1 \\ s\theta_1 & c\theta_1 & 0 & a_1 s\theta_1 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.9}
$$

TABLE 2.2. D-H Parameters of the SCARA Arm

Joint i	α_i	a_i	d_i	θ_i
1	0	a_1	d_1	θ_1
2	π	a_2	0	θ_2
3	0	0	d_3	0
4	0	0	d_4	θ_4

$$
{}^1A_2 = \begin{bmatrix} c\theta \\ s\theta \\ 0 \\ 0 \end{bmatrix}
$$

$$
{}^2A_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
{}^3A_4 = \begin{bmatrix} c\theta_4 \\ s\theta_4 \\ 0 \\ 0 \end{bmatrix}
$$

In this robot, the joint variables
ables control the x and y coor
z coordinate, and the fourth join
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end effector cannot be specified
must be always pointing in the
robot has only 4 degrees of freed
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2.4 LOOP-CLOSURE EQU

In a study of the kinematics of ro
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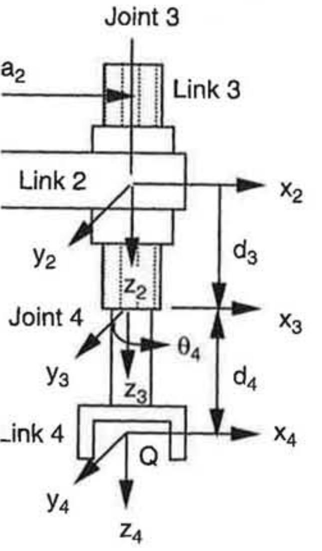
where the upper right 3×1 sub
point Q and the upper left $3 \times$
end effector. The orientation of t
three Euler angles, or the directi
nate axes, u, v, and w. If the w–
elements of the upper left 3×3

$$ u_x = c $$

$$ u_y = s $$



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A arm is constructed with four
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2.4 shows a schematic diagram
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Eq. (2.2), we obtain the D-H

$$a_1c\theta_1 \quad a_1s\theta_1 \quad d_1 \quad 1 \qquad (2.9)$$

CARA Arm

d_i	θ_i
d_1	θ_1
0	θ_2
d_3	0
d_4	θ_4

$$^1A_2 = \begin{bmatrix} c\theta_2 & s\theta_2 & 0 & a_2c\theta_2 \\ s\theta_2 & -c\theta_2 & 0 & a_2s\theta_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.10}$$

$$^2A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.11}$$

$$^3A_4 = \begin{bmatrix} c\theta_4 & -s\theta_4 & 0 & 0 \\ s\theta_4 & c\theta_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.12}$$

In this robot, the joint variables are θ_1, θ_2, d_3, and θ_4. The first two joint variables control the x and y coordinates, the third joint variable controls the z coordinate, and the fourth joint variable controls the orientation of the end effector. Since the robot has only 4 degrees of freedom, the orientation of the end effector cannot be specified arbitrarily. As a matter of fact, the z_4-axis must be always pointing in the negative z_0 direction. Although the SCARA robot has only 4 degrees of freedom, it is very useful for assembling components on a plane such as a PC board.

2.4 LOOP-CLOSURE EQUATIONS

In a study of the kinematics of robot manipulators, we are interested in deriving an algebraic equation relating the location of the end effector to the joint variables. The location of the end effector can be specified by the following 4×4 homogeneous transformation matrix:

$$^0A_n = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} & \mathbf{q} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.13}$$

where the upper right 3×1 submatrix describes the position of a reference point Q and the upper left 3×3 submatrix describes the orientation of the end effector. The orientation of the end effector can be specified in terms of three Euler angles, or the direction cosines of the three end-effector coordinate axes, u, v, and w. If the w–u–w Euler angles are used, for example, the elements of the upper left 3×3 submatrix are given by

$$u_x = c\phi c\psi - s\phi c\theta s\psi,$$
$$u_y = s\phi c\psi + c\phi c\theta s\psi,$$

$$\begin{aligned}
u_z &= s\theta s\psi, \\
v_x &= -c\phi s\psi - s\phi c\theta c\psi, \\
v_y &= -s\phi s\psi + c\phi c\theta c\psi, \\
v_z &= s\theta c\psi, \\
w_x &= s\phi s\theta, \\
w_y &= -c\phi s\theta, \\
w_z &= c\theta. \qquad (2.14)
\end{aligned}$$

If the direction cosines are used, **u**, **v**, and **w** represent three unit vectors directed along the three coordinate axes of the hand coordinate system and expressed in the base coordinate system.

From the geometry of the links, the transformation matrix ${}^0A_n$ above can be thought of as the resultant of a series of coordinate transformations beginning from the base coordinate system to the end-effector coordinate system. That is,

$$ {}^0A_1\,{}^1A_2\,{}^2A_3 \cdots {}^{n-1}A_n = {}^0A_n. \qquad (2.15)$$

Equation (2.15) is called the *loop-closure equation* of a serial manipulator. It contains 16 scalar equations, four of which are trivial. Equating the upper right 3×1 submatrix results in three independent equations, representing the position of the end effector. Equating the elements of the upper left 3×3 submatrix results in nine equations, representing the orientation of the end effector. However, only three of the nine orientation equations are independent because of the orthogonal conditions.

The loop-closure equation, Eq. (2.15), can be used to solve both direct and inverse kinematics problems. For direct kinematics, the joint variables are given and the problem is to find where the end effector is with respect to the base coordinate system. This can be accomplished by multiplying the D-H matrices on the left-hand side of the equation. For the inverse kinematics, the end-effector location (i.e., ${}^0A_n$) is given and the problem is to find the joint variables needed to bring the end effector to the desired location. The problem becomes very nonlinear. In what follows, we concentrate on the inverse kinematics problem.

Example 2.4.1 *Scorbot Robot* Figure 2.5 shows a schematic diagram of the Scorbot robot. In this diagram, the second, third, and fourth joint axes are parallel to one another and point into the paper at points A, B, and P, respectively. The first joint axis points up vertically, and the fifth joint axis intersects the fourth perpendicularly. We wish to find the overall transformation matrix for the robot.

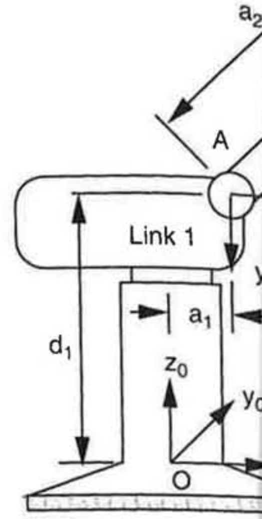

FIGURE 2

Using the coordinate
link parameters are liste
into Eq. (2.2), we obtai

${}^0A_1$

TABLE 2.3

Joint i
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3
4
5

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(2.14)

represent three unit vectors
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${}^0A_n$. (2.15)

cation of a serial manipulator.
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used to solve both direct and
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effector is with respect to the
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or the inverse kinematics, the
problem is to find the joint
desired location. The prob-
ve concentrate on the inverse

ows a schematic diagram of
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t points A, B, and P, respec-
d the fifth joint axis intersects
overall transformation matrix

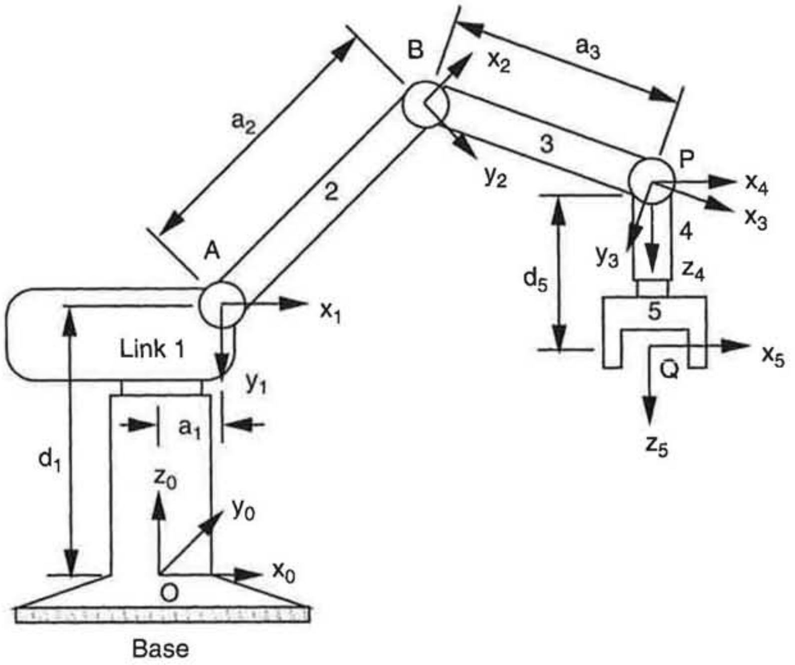

FIGURE 2.5. Schematic diagram of the Scorbot robot.

Using the coordinate systems established in Fig. 2.5, the corresponding link parameters are listed in Table 2.3. Substituting the D-H link parameters into Eq. (2.2), we obtain the D-H transformation matrices:

$$
{}^0A_1 = \begin{bmatrix} c\theta_1 & 0 & -s\theta_1 & a_1 c\theta_1 \\ s\theta_1 & 0 & c\theta_1 & a_1 s\theta_1 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad (2.16)
$$

TABLE 2.3. D-H Parameters of a 5-DOF Manipulator

Joint i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	a_1	d_1	θ_1
2	0	a_2	0	θ_2
3	0	a_3	0	θ_3
4	$-\pi/2$	0	0	θ_4
5	0	0	d_5	θ_5

$$
{}^{1}A_2 = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & a_2 c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & a_2 s\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.17}
$$

$$
{}^{2}A_3 = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_3 c\theta_3 \\ s\theta_3 & c\theta_3 & 0 & a_3 s\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.18}
$$

$$
{}^{3}A_4 = \begin{bmatrix} c\theta_4 & 0 & -s\theta_4 & 0 \\ s\theta_4 & 0 & c\theta_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.19}
$$

$$
{}^{4}A_5 = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & 0 \\ s\theta_5 & c\theta_5 & 0 & 0 \\ 0 & 0 & 1 & d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.20}
$$

Multiplying Eqs. (2.17), (2.18), and (2.19) yields

$$
{}^{1}A_4 = \begin{bmatrix} c\theta_{234} & 0 & -s\theta_{234} & a_3 c\theta_{23} + a_2 c\theta_2 \\ s\theta_{234} & 0 & c\theta_{234} & a_3 s\theta_{23} + a_2 s\theta_2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.21}
$$

where $c\theta_{ij} = \cos(\theta_i + \theta_j)$, $s\theta_{ij} = \sin(\theta_i + \theta_j)$, $c\theta_{ijk} = \cos(\theta_i + \theta_j + \theta_k)$, and $s\theta_{ijk} = \sin(\theta_i + \theta_j + \theta_k)$.

Note that Eq. (2.21) provides a transformation from the fourth coordinate system to the first coordinate system. We may treat θ_2, θ_{23}, and θ_{234} as new variables. In this way, the orientation submatrix contains only one variable, θ_{234}, while the position submatrix contains two variables, θ_2 and θ_{23}. This important fact can be used for deriving a closed-form solution for any manipulator with three consecutive parallel joint axes.

Multiplying Eqs. (2.16), (2.21), and (2.20) yields the elements of the overall transformation matrix ${}^{0}A_5$:

$$
\begin{aligned}
u_x &= c\theta_1 c\theta_{234} c\theta_5 + s\theta_1 s\theta_5, \\
u_y &= s\theta_1 c\theta_{234} c\theta_5 - c\theta_1 s\theta_5, \\
u_z &= -s\theta_{234} c\theta_5, \\
v_x &= -c\theta_1 c\theta_{234} s\theta_5 + s\theta_1 c\theta_5,
\end{aligned}
$$

$v_y = -s\theta_1$

$v_z = s\theta_{234}$

$w_x = -c\theta$

$w_y = -s\theta$

$w_z = -c\theta$

$q_x = c\theta_1($

$q_y = s\theta_1($

$q_z = d_1 -$

Since this is a 5-dof man
effector can be specified.
direction of a line in the e
direction of x_5-axis) are sp
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2.5 OTHER COORDIN

In the preceding section, th
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another coordinate system
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where ${}^{\text{ref}}A_0$ and ${}^{n}A_{\text{tool}}$ are

2.6 DENAVIT–HART

Although the loop-closu
inverse kinematics proble
In general, if there are t
position of the point of i
associated with the thre
axes, we may combine t

$$
\begin{aligned}
v_y &= -s\theta_1 c\theta_{234} s\theta_5 - c\theta_1 c\theta_5, \\
v_z &= s\theta_{234} s\theta_5, \\
w_x &= -c\theta_1 s\theta_{234}, \\
w_y &= -s\theta_1 s\theta_{234}, \\
w_z &= -c\theta_{234}, \\
q_x &= c\theta_1 (a_1 + a_2 c\theta_2 + a_3 c\theta_{23} - d_5 s\theta_{234}), \\
q_y &= s\theta_1 (a_1 + a_2 c\theta_2 + a_3 c\theta_{23} - d_5 s\theta_{234}), \\
q_z &= d_1 - a_2 s\theta_2 - a_3 s\theta_{23} - d_5 c\theta_{234}.
\end{aligned} \tag{2.22}
$$

Since this is a 5-dof manipulator, only five of the six parameters of the end effector can be specified. Very often, the desired position of a point and the direction of a line in the end effector (e.g., the position of point Q and the direction of x_5-axis) are specified. Five-dof manipulators are useful for spray painting, spot welding, and sealant applications for which only the position and direction of a line are essential.

2.5 OTHER COORDINATE SYSTEMS

In the preceding section, the z_0-axis of the base coordinate system was chosen to be in line with the first joint axis, and the z_n-axis of the hand coordinate system was chosen to be in the direction of approach. If an additional coordinate system is defined in the base with a transformation matrix ${}^{\text{ref}}A_0$, and another coordinate system is defined in the tool frame with a transformation matrix ${}^{n}A_{\text{tool}}$, the overall loop-closure equation can be modified as

$$
{}^{\text{ref}}A_{\text{tool}} = {}^{\text{ref}}A_0 \, {}^{0}A_n \, {}^{n}A_{\text{tool}}, \tag{2.23}
$$

where ${}^{\text{ref}}A_0$ and ${}^{n}A_{\text{tool}}$ are constant transformation matrices.

2.6 DENAVIT–HARTENBERG METHOD

Although the loop-closure equation, Eq. (2.15), can be applied to solve the inverse kinematics problem, in practice it is rarely solved in its present form. In general, if there are three intersecting joint axes, we may work with the position of the point of intersection first, thereby avoiding the joint variables associated with the three intersecting axes. If there are three parallel joint axes, we may combine the three joint variables as illustrated in the Scorbot robot example. We may also pre- or postmultiply the loop-closure equation by the inverse of the matrix ${}^{i-1}A_i$ to obtain alternative loop-closure equations, such as

$$({}^0A_1)^{-1}\,{}^0A_n = {}^1A_2\,{}^2A_3 \cdots {}^{n-1}A_n, \tag{2.24}$$

$$({}^1A_2)^{-1}({}^0A_1)^{-1}\,{}^0A_n = {}^2A_3\,{}^3A_4 \cdots {}^{n-1}A_n, \tag{2.25}$$

$$({}^2A_3)^{-1}({}^1A_2)^{-1}({}^0A_1)^{-1}\,{}^0A_n = {}^3A_4\,{}^4A_5 \cdots {}^{n-1}A_n. \tag{2.26}$$

One reason for rearranging the loop-closure equation is to redistribute the unknown variables on both sides of the equation as evenly as possible. Another reason is to take advantage of some special conditions, such as three consecutive intersecting joint axes or three consecutive parallel joint axes. In many cases, the equation becomes decoupled and a closed-form solution can be derived.

2.6.1 Position Analysis of a Planar 3-DOF Manipulator

For the planar 3-dof manipulator shown in Fig. 2.3, the overall transformation matrix is given by

$${}^0A_3 = {}^0A_1\,{}^1A_2\,{}^2A_3. \tag{2.27}$$

Substituting Eqs. (2.6) through (2.8) into (2.27), we obtain

$${}^0A_3 = \begin{bmatrix} c\theta_{123} & -s\theta_{123} & 0 & a_1c\theta_1 + a_2c\theta_{12} + a_3c\theta_{123} \\ s\theta_{123} & c\theta_{123} & 0 & a_1s\theta_1 + a_2s\theta_{12} + a_3s\theta_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.28}$$

(a) Direct Kinematics. The position vector of the origin Q expressed in the end-effector coordinate system is given by ${}^3\mathbf{q} = [0, 0, 0, 1]^T$. Let the position vector of Q with respect to the base coordinate system be ${}^0\mathbf{q} = [q_x, q_y, q_z, 1]^T$. Then we can relate ${}^3\mathbf{q}$ to ${}^0\mathbf{q}$ by the following transformation:

$$\begin{bmatrix} q_x \\ q_y \\ q_z \\ 1 \end{bmatrix} = {}^0A_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1c\theta_1 + a_2c\theta_{12} + a_3c\theta_{123} \\ a_1s\theta_1 + a_2s\theta_{12} + a_3s\theta_{123} \\ 0 \\ 1 \end{bmatrix}. \tag{2.29}$$

Hence, given θ_1, θ_2, and θ_3, the position of point Q can be computed by Eq. (2.29). Similarly, the position vector of any other point in the end effector,

${}^3\mathbf{g} = [g_u, g_v, 0, 1]^T$, is given by

$$\begin{bmatrix} g_x \\ g_y \\ g_z \\ 1 \end{bmatrix} = {}^0A_3 \begin{bmatrix} g_u \\ g_v \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} g_u c\theta_{123} - g_v s\theta_{123} + a_1 c\theta_1 + a_2 c\theta_{12} + a_3 c\theta_{123} \\ g_u s\theta_{123} + g_v c\theta_{123} + a_1 s\theta_1 + a_2 s\theta_{12} + a_3 s\theta_{123} \\ 0 \\ 1 \end{bmatrix}. \tag{2.30}$$

From Eq. (2.28), we conclude that the orientation angle of the end effector is equal to $\theta_1 + \theta_2 + \theta_3$.

(b) Inverse Kinematics. For the inverse kinematics problem, the location of the end effector is given and the problem is to find the joint angles θ_i, $i = 1, 2, 3$, necessary to bring the end effector to the desired location. For a planar 3-dof manipulator, the end effector can be specified in terms of the position of point Q and an orientation angle ϕ of the end effector. Hence the overall transformation matrix from the end-effector coordinate system to the base coordinate system, ${}^0A_3$, is given by

$${}^0A_3 = \begin{bmatrix} c\phi & -s\phi & 0 & q_x \\ s\phi & c\phi & 0 & q_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.31}$$

Inverse kinematics solutions can be obtained by equating the elements of Eq. (2.28) to that of (2.31). To find the orientation of the end effector, we equate the (1,1) and (2,1) elements of Eq. (2.28) to that of (2.31):

$$c\theta_{123} = c\phi, \tag{2.32}$$

$$s\theta_{123} = s\phi. \tag{2.33}$$

Hence

$$\theta_{123} = \theta_1 + \theta_2 + \theta_3 = \phi. \tag{2.34}$$

Next, we equate the (1,4) and (2,4) elements of Eq. (2.28) to that of (2.31):

$$p_x = a_1 c\theta_1 + a_2 c\theta_{12}, \tag{2.35}$$

$$p_y = a_1 s\theta_1 + a_2 s\theta_{12}, \tag{2.36}$$

where $p_x = q_x - a_3 c\phi$ and $p_y = q_y - a_3 s\phi$ denote the position vector of the point P located at the third joint axis shown in Fig. 2.3. Note that by using this substitution, θ_3 disappears from Eqs. (2.35) and (2.36). From Fig. 2.3 we observe that the distance from point O to P is independent of θ_1. Hence we can eliminate θ_1 by summing the squares of Eqs. (2.35) and (2.36); that is,

$$p_x^2 + p_y^2 = a_1^2 + a_2^2 + 2a_1a_2c\theta_2. \tag{2.37}$$

Solving Eq. (2.37) for θ_2, we obtain

$$\theta_2 = \cos^{-1}\kappa, \tag{2.38}$$

where

$$\kappa = \frac{p_x^2 + p_y^2 - a_1^2 - a_2^2}{2a_1a_2}.$$

Equation (2.38) yields (1) two real roots if $|\kappa| < 1$, (2) one double root if $|\kappa| = 1$, and (3) no real roots if $|\kappa| > 1$. In general, if $\theta_2 = \theta_2^*$ is a solution, $\theta_2 = -\theta_2^*$ is also a solution, where $\pi \geq \theta_2^* \geq 0$. We call $\theta_2 = \theta_2^*$ the elbow-down solution and $\theta_2 = -\theta_2^*$ the elbow-up solution. If $|\kappa| = 1$, the arm is in a fully stretched or folded configuration. If $|\kappa| > 1$, the position is not reachable.

Corresponding to each θ_2, we can solve θ_1 by expanding Eqs. (2.35) and (2.36) as follows:

$$(a_1 + a_2c\theta_2)c\theta_1 - (a_2s\theta_2)s\theta_1 = p_x, \tag{2.39}$$

$$(a_2s\theta_2)c\theta_1 + (a_1 + a_2c\theta_2)s\theta_1 = p_y. \tag{2.40}$$

Solving Eqs. (2.39) and (2.40) for $c\theta_1$ and $s\theta_1$, yields

$$c\theta_1 = \frac{p_x(a_1 + a_2c\theta_2) + p_ya_2s\theta_2}{\Delta},$$

$$s\theta_1 = \frac{-p_xa_2s\theta_2 + p_y(a_1 + a_2c\theta_2)}{\Delta}.$$

where $\Delta = a_1^2 + a_2^2 + 2a_1a_2c\theta_2$. Hence, corresponding to each θ_2, we obtain a unique solution for θ_1:

$$\theta_1 = \text{Atan2}(s\theta_1, c\theta_1). \tag{2.41}$$

In a computer program we may use the function Atan2(x, y) to obtain a unique solution for θ_1. However, the solution may be real or complex. A complex solution corresponds to an end-effector location that is not reachable by the manipulator. Once θ_1 and θ_2 are known, Eq. (2.34) yields a unique solution for θ_3. Hence, corresponding to a given end-effector location, there are

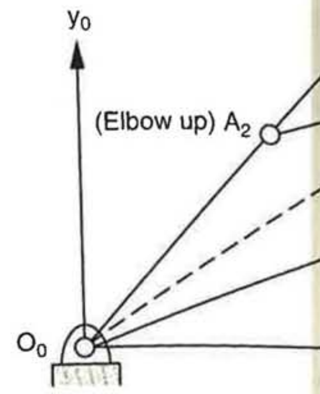

FIGURE 2.6. Two p

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Vector-Loop Method. Althoug
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$$_1 a_2 c\theta_2. \quad (2.37)$$

(2.38)

$$\frac{a_2^2}{}.$$

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$$_1 = p_x, \quad (2.39)$$

$$_1 = p_y. \quad (2.40)$$

ields

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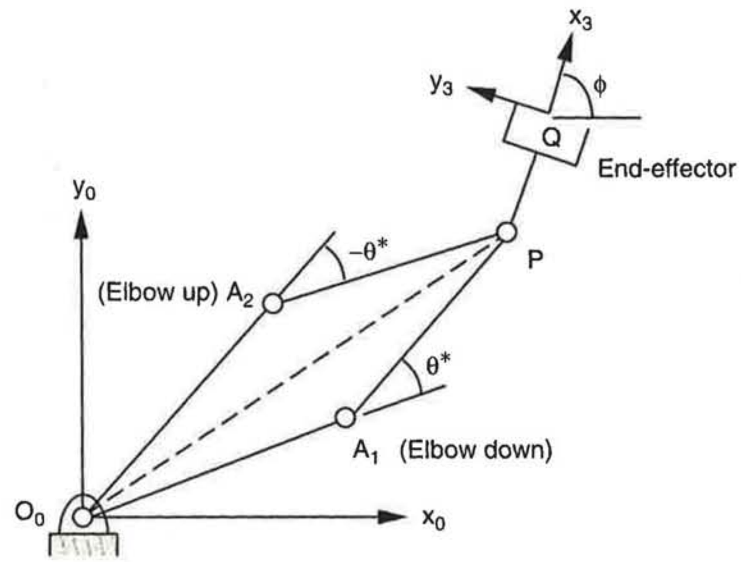

FIGURE 2.6. Two possible inverse kinematics solutions.

generally two real inverse kinematics solutions, one being the reflection of the other about a line connecting points O and P, as illustrated in Fig. 2.6.

Vector-Loop Method. Although the D-H method of analysis is a very powerful tool, the inverse kinematics problem can often be solved by other methods, such as the vector-loop method. For example, the vector-loop method becomes more efficient for analysis of the 3-dof planar manipulator shown in Fig. 2.3. For convenience, the manipulator has been resketched as shown in Fig. 2.7.

Using vector algebra, the position vector of the wrist center P can be related to the origin Q of the end effector by the equations

$$p_x = q_x - a_3 c\phi, \quad (2.42)$$

$$p_y = q_y - a_3 s\phi. \quad (2.43)$$

From Fig. 2.7, we observe that the orientation angle ϕ is related to the joint angles by

$$\phi = \theta_1 + \theta_2 + \theta_3. \quad (2.44)$$

We now form a fictitious vector loop equation as follows:

$$\overline{OA} + \overline{AP} = \overline{OP}. \quad (2.45)$$

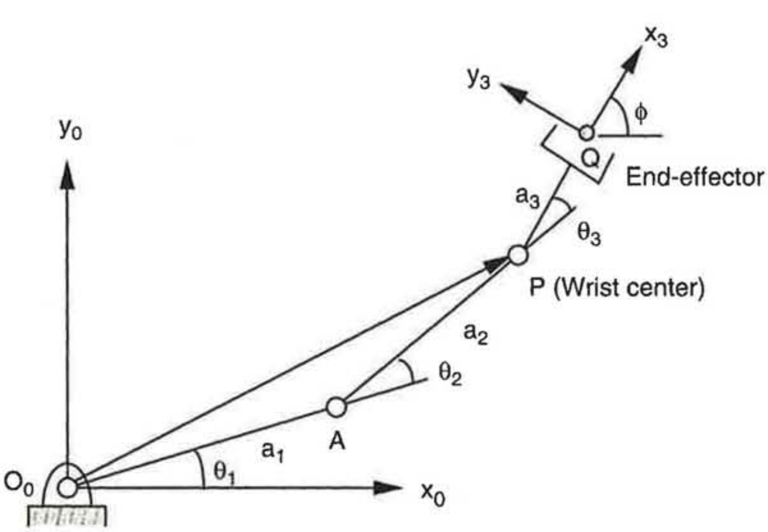

FIGURE 2.7. Vector loop of the planar 3-dof manipulator.

Taking the x and y components of Eq. (2.45) yields

$$p_x = a_1 c\theta_1 + a_2 c\theta_{12}, \tag{2.46}$$

$$p_y = a_1 s\theta_1 + a_2 s\theta_{12}. \tag{2.47}$$

Note that using the vector-loop method, we have derived Eqs. (2.35) and (2.36) with very little effort.

2.6.2 Position Analysis of the Scorbot Robot

For the Scorbot robot shown in Fig. 2.5, the overall transformation matrix is given by Eq. (2.22). We wish to solve the direct and inverse kinematics problems.

(a) Direct Kinematics. For the direct kinematics problem, we simply substitute the given joint angles into Eq. (2.22) to obtain the end-effector position, (q_x, q_y, q_z), and the orientation in terms of the three unit vectors (u_x, u_y, u_z), (v_x, v_y, v_z), and (w_x, w_y, w_z).

(b) Inverse Kinematics. For the inverse kinematics problem, only 5 of the 12 parameters associated with the end-effector position vector and rotation matrix can be specified at will. This is because the manipulator has only 5 degrees of freedom. It is obvious that the position vector **q** and the approach

vector **w** cannot be specified
together depend only on 4 d
exercise we assume that **q** a
vectors, **v** and **w**, are to be de
Although Eq. (2.22) can b
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of the loop-closure equation
$({}^0A_1)$
Equating the first column
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$q_x c\theta_1 + q_y s\theta_1$
$-q_z$
$-q_x s\theta_1 +$
The first joint angle, θ_1, is
There are two solutions; tha
solution. Once θ_1 is found, t
θ_5
That is, if $\theta_5 = \theta_5^*$ is a soluti
Corresponding to each s
duce a unique solution of θ_2
$\theta_{234} = \text{Atan2}$
Next, we solve Eqs. (2.52)
(2.53) can be written

vector **w** cannot be specified simultaneously, due to the fact that **q** and **w** together depend only on 4 degrees of freedom of the manipulator. For this exercise we assume that **q** and **u** are specified and that the other two unit vectors, **v** and **w**, are to be determined after the joint angles are found.

Although Eq. (2.22) can be used to solve the inverse kinematics, in what follows we take a more straightforward approach by multiplying both sides of the loop-closure equation by $({}^0A_1)^{-1}$; that is,

$$({}^0A_1)^{-1}\,{}^0A_5 = {}^1A_2\,{}^2A_3\,{}^3A_4\,{}^4A_5. \tag{2.48}$$

Equating the first column of Eq. (2.48), we obtain

$$u_x c\theta_1 + u_y s\theta_1 = c\theta_{234} c\theta_5, \tag{2.49}$$

$$-u_z = s\theta_{234} c\theta_5, \tag{2.50}$$

$$-u_x s\theta_1 + u_y c\theta_1 = -s\theta_5. \tag{2.51}$$

Similarly, equating the fourth column of Eq. (2.48), we obtain

$$q_x c\theta_1 + q_y s\theta_1 - a_1 = a_2 c\theta_2 + a_3 c\theta_{23} - d_5 s\theta_{234}, \tag{2.52}$$

$$-q_z + d_1 = a_2 s\theta_2 + a_3 s\theta_{23} + d_5 c\theta_{234}, \tag{2.53}$$

$$-q_x s\theta_1 + q_y c\theta_1 = 0. \tag{2.54}$$

The first joint angle, θ_1, is obtained immediately from Eq. (2.54):

$$\theta_1 = \tan^{-1}\frac{q_y}{q_x}. \tag{2.55}$$

There are two solutions; that is, if $\theta_1 = \theta_1^*$ is a solution, $\theta_1 = \pi + \theta_1^*$ is also a solution. Once θ_1 is found, two solutions for θ_5 are obtained from Eq. (2.51):

$$\theta_5 = \sin^{-1}(u_x s\theta_1 - u_y c\theta_1). \tag{2.56}$$

That is, if $\theta_5 = \theta_5^*$ is a solution, $\theta_5 = \pi - \theta_5^*$ is also a solution.

Corresponding to each solution set of (θ_1, θ_5), Eqs. (2.49) and (2.50) produce a unique solution of θ_{234}:

$$\theta_{234} = \text{Atan2}\left[-u_z/c\theta_5,\ (u_x c\theta_1 + u_y s\theta_1)/c\theta_5\right]. \tag{2.57}$$

Next, we solve Eqs. (2.52) and (2.53) for θ_2 and θ_3. Equations (2.52) and (2.53) can be written

$$a_2 c\theta_2 + a_3 c\theta_{23} = k_1, \tag{2.58}$$

$$a_2 s\theta_2 + a_3 s\theta_{23} = k_2. \tag{2.59}$$

where $k_1 = q_x c\theta_1 + q_y s\theta_1 - a_1 + d_5 s\theta_{234}$ and $k_2 = -q_z + d_1 - d_5 c\theta_{234}$.

Summing the squares of Eqs. (2.58) and (2.59) yields

$$a_2^2 + a_3^2 + 2a_2 a_3 c\theta_3 = k_1^2 + k_2^2. \tag{2.60}$$

Hence

$$\theta_3 = \cos^{-1}\frac{k_1^2 + k_2^2 - a_2^2 - a_3^2}{2a_2 a_3}. \tag{2.61}$$

and there are two solutions of θ_3; that is, if $\theta_3 = \theta_3^*$ is a solution, $\theta_3 = -\theta_3^*$ is also a solution.

Once θ_3 is known, we can solve θ_2 by expanding Eqs. (2.58) and (2.59) as follows:

$$(a_2 + a_3 c\theta_3)c\theta_2 - (a_3 s\theta_3)s\theta_2 = k_1, \tag{2.62}$$

$$(a_3 s\theta_3)c\theta_2 + (a_2 + a_3 c\theta_3)s\theta_2 = k_2. \tag{2.63}$$

Solving Eqs. (2.62) and (2.63) for $c\theta_2$ and $s\theta_2$ yields

$$c\theta_2 = \frac{k_1(a_2 + a_3 c\theta_3) + k_2 a_3 s\theta_3}{a_2^2 + a_3^2 + 2a_2 a_3 c\theta_3},$$

$$s\theta_2 = \frac{-k_1 a_3 s\theta_3 + k_2(a_2 + a_3 c\theta_3)}{a_2^2 + a_3^2 + 2a_2 a_3 c\theta_3}.$$

Hence, corresponding to each solution set of $(\theta_1, \theta_3, \theta_5, \theta_{234})$, we obtain a unique solution of θ_2:

$$\theta_2 = \mathrm{Atan2}(s\theta_2, c\theta_2). \tag{2.64}$$

Finally, θ_4 is obtained by

$$\theta_4 = \theta_{234} - \theta_2 - \theta_3. \tag{2.65}$$

We conclude that corresponding to each given end-effector location, there are at most eight inverse kinematics solutions.

2.6.3 Position Analysi

Figure 2.8 shows a 6-dof ma
ulator, the first joint axis p
joint axis is perpendicular t
$a_1 = OA$, the third joint
tance $a_2 = AB$, and the fo
axis with a small offset dis
axes intersect one another p
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The kinematics problem fo
two subchains: one associa
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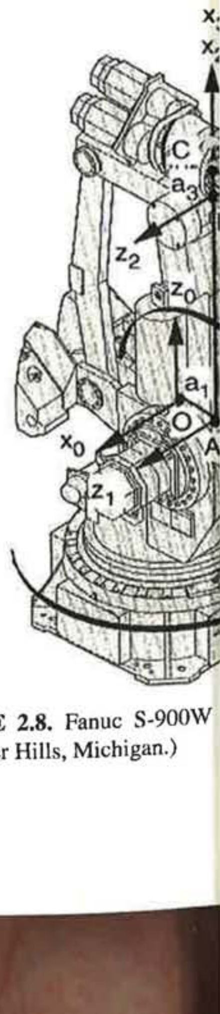

FIGURE 2.8. Fanuc S-900W
Rochester Hills, Michigan.)

$\ldots_1,$ (2.58)

$\ldots_2.$ (2.59)

$k_2 = -q_z + d_1 - d_5 c\theta_{234}.$

59) yields

$\ldots_1^2 + k_2^2.$ (2.60)

$\ldots_2^2 - a_3^2$ (2.61)

$= \theta_3^*$ is a solution, $\theta_3 = -\theta_3^*$ is

nding Eqs. (2.58) and (2.59) as

$\ldots s\theta_2 = k_1,$ (2.62)

$\ldots s\theta_2 = k_2.$ (2.63)

yields

$\ldots k_2 a_3 s\theta_3 / \ldots_3 c\theta_3,$

$\ldots + a_3 c\theta_3) / \ldots a_3 c\theta_3.$

f $(\theta_1, \theta_3, \theta_5, \theta_{234})$, we obtain a

$\ldots \theta_2).$ (2.64)

$\theta_3.$ (2.65)

ven end-effector location, there

2.6.3 Position Analysis of the Fanuc S-900W Robot

Figure 2.8 shows a 6-dof manipulator manufactured by Fanuc. In this manipulator, the first joint axis points up vertically along the z_0-axis, the second joint axis is perpendicular to the first joint axis with a small offset distance $a_1 = OA$, the third joint axis is parallel to the second with an offset distance $a_2 = AB$, and the fourth joint axis is perpendicular to the third joint axis with a small offset distance $a_3 = BC$. In addition, the last three joint axes intersect one another perpendicularly in sequence at a common point P, which is d_4 distance away from point C. This robot belongs to a special class of manipulators where the last three joint axes intersect at the wrist center. The kinematics problem for this type of manipulators can be partitioned into two subchains: one associated with the first three moving links and the other with the last three moving links. That is, in solving the inverse kinematics problem, the position of the wrist center can be solved independently of the orientation part, therefore reducing the complexity of the problem.

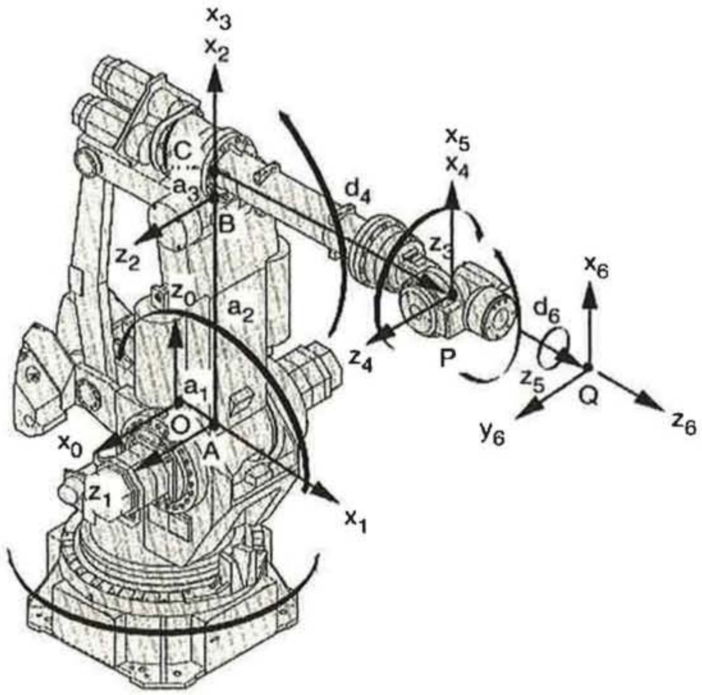

FIGURE 2.8. Fanuc S-900W robot. (Courtesy of Fanuc Robotics North America, Inc., Rochester Hills, Michigan.)

TABLE 2.4. D-H Parameters of the Fanuc S-900W Manipulator

Joint i	α_i	a_i	d_i	θ_i
1	$\pi/2$	a_1	0	θ_1
2	0	a_2	0	θ_2
3	$\pi/2$	a_3	0	θ_3
4	$-\pi/2$	0	d_4	θ_4
5	$\pi/2$	0	0	θ_5
6	0	0	d_6	θ_6

We note that this manipulator employs a four-bar linkage to drive the third joint. The four-bar linkage simply transmits the motion of the third motor mounted on the waist to the third joint. Otherwise, it has no effect on the kinematics of the manipulator. In the following analysis we neglect the effect of the four-bar linkage and treat the manipulator as a serial manipulator.

Using the coordinate systems established in Fig. 2.8, the corresponding link parameters are listed in Table 2.4. Substituting the D-H link parameters into Eq. (2.2), we obtain the D-H transformation matrices:

$$ {}^0A_1 = \begin{bmatrix} c\theta_1 & 0 & s\theta_1 & a_1c\theta_1 \\ s\theta_1 & 0 & -c\theta_1 & a_1s\theta_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.66} $$

$$ {}^1A_2 = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & a_2c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & a_2s\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.67} $$

$$ {}^2A_3 = \begin{bmatrix} c\theta_3 & 0 & s\theta_3 & a_3c\theta_3 \\ s\theta_3 & 0 & -c\theta_3 & a_3s\theta_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.68} $$

$$ {}^3A_4 = \begin{bmatrix} c\theta_4 & 0 & -s\theta_4 & 0 \\ s\theta_4 & 0 & c\theta_4 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.69} $$

$$ {}^4A_5 = \begin{bmatrix} c\theta_5 & 0 & s\theta_5 & 0 \\ s\theta_5 & 0 & -c\theta_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.70} $$

$$
{}^{5}A_6 = \begin{bmatrix} c\theta_6 & -s\theta_6 & 0 & 0 \\ s\theta_6 & c\theta_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.71}
$$

The end-effector location is given by

$$
{}^{0}A_6 = \begin{bmatrix} u_x & v_x & w_x & q_x \\ u_y & v_y & w_y & q_y \\ u_z & v_z & w_z & q_z \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.72}
$$

The loop-closure equation is obtained in two steps. First, we multiply Eqs. (2.66), (2.67), and (2.68):

$$
\begin{aligned}
{}^{0}A_3 &= {}^{0}A_1\,{}^{1}A_2\,{}^{2}A_3 \\
&= \begin{bmatrix} c\theta_1 c\theta_{23} & s\theta_1 & c\theta_1 s\theta_{23} & c\theta_1(a_1 + a_2 c\theta_2 + a_3 c\theta_{23}) \\ s\theta_1 c\theta_{23} & -c\theta_1 & s\theta_1 s\theta_{23} & s\theta_1(a_1 + a_2 c\theta_2 + a_3 c\theta_{23}) \\ s\theta_{23} & 0 & -c\theta_{23} & a_2 s\theta_2 + a_3 s\theta_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned} \tag{2.73}
$$

Next, we multiply Eqs. (2.69), (2.70), and (2.71):

$$
\begin{aligned}
{}^{3}A_6 &= {}^{3}A_4\,{}^{4}A_5\,{}^{5}A_6 \\
&= \begin{bmatrix} c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6 & -c\theta_4 c\theta_5 s\theta_6 - s\theta_4 c\theta_6 & c\theta_4 s\theta_5 & d_6 c\theta_4 s\theta_5 \\ s\theta_4 c\theta_5 c\theta_6 + c\theta_4 s\theta_6 & -s\theta_4 c\theta_5 s\theta_6 + c\theta_4 c\theta_6 & s\theta_4 s\theta_5 & d_6 s\theta_4 s\theta_5 \\ -s\theta_5 c\theta_6 & s\theta_5 s\theta_6 & c\theta_5 & d_4 + d_6 c\theta_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned} \tag{2.74}
$$

Hence the resulting transformation matrix is given by

$$
{}^{0}A_6 = {}^{0}A_3\,{}^{3}A_6, \tag{2.75}
$$

where ${}^{0}A_6$ describes the end effector location.

Substituting Eqs. (2.73) and (2.74) into (2.75) yields the elements of ${}^{0}A_6$ as follows:

$$
\begin{aligned}
u_x &= c\theta_1[c\theta_{23}(c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6) - s\theta_{23} s\theta_5 c\theta_6] + s\theta_1(s\theta_4 c\theta_5 c\theta_6 + c\theta_4 s\theta_6), \\
u_y &= s\theta_1[c\theta_{23}(c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6) - s\theta_{23} s\theta_5 c\theta_6] - c\theta_1(s\theta_4 c\theta_5 c\theta_6 + c\theta_4 s\theta_6), \\
u_z &= s\theta_{23}(c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6) + c\theta_{23} s\theta_5 c\theta_6,
\end{aligned}
$$

$$
\begin{aligned}
v_x &= c\theta_1[-c\theta_{23}(c\theta_4 c\theta_5 s\theta_6 + s\theta_4 c\theta_6) + s\theta_{23} s\theta_5 s\theta_6] \\
&\quad + s\theta_1(-s\theta_4 c\theta_5 s\theta_6 + c\theta_4 c\theta_6), \\
v_y &= s\theta_1[-c\theta_{23}(c\theta_4 c\theta_5 s\theta_6 + s\theta_4 c\theta_6) + s\theta_{23} s\theta_5 s\theta_6] \\
&\quad - c\theta_1(-s\theta_4 c\theta_5 s\theta_6 + c\theta_4 c\theta_6), \\
v_z &= -s\theta_{23}(c\theta_4 c\theta_5 s\theta_6 + s\theta_4 c\theta_6) - c\theta_{23} s\theta_5 s\theta_6, \\
w_x &= c\theta_1(c\theta_{23} c\theta_4 s\theta_5 + s\theta_{23} c\theta_5) + s\theta_1 s\theta_4 s\theta_5, \\
w_y &= s\theta_1(c\theta_{23} c\theta_4 s\theta_5 + s\theta_{23} c\theta_5) - c\theta_1 s\theta_4 s\theta_5, \\
w_z &= s\theta_{23} c\theta_4 s\theta_5 - c\theta_{23} c\theta_5, \\
q_x &= c\theta_1[a_1 + a_2 c\theta_2 + a_3 c\theta_{23} + d_4 s\theta_{23} + d_6(c\theta_{23} c\theta_4 s\theta_5 + s\theta_{23} c\theta_5)] \\
&\quad + d_6 s\theta_1 s\theta_4 s\theta_5, \\
q_y &= s\theta_1[a_1 + a_2 c\theta_2 + a_3 c\theta_{23} + d_4 s\theta_{23} + d_6(c\theta_{23} c\theta_4 s\theta_5 + s\theta_{23} c\theta_5)] \\
&\quad - d_6 c\theta_1 s\theta_4 s\theta_5, \\
q_z &= a_2 s\theta_2 + a_3 s\theta_{23} - d_4 c\theta_{23} + d_6(s\theta_{23} c\theta_4 s\theta_5 - c\theta_{23} c\theta_5).
\end{aligned}
$$

Although the equations above can be used to solve the inverse kinematics, they are highly nonlinear and difficult to solve. In what follows we present a more efficient method of solution by separating the wrist-center-position problem from the orientation problem.

(a) Wrist Center Position. Note that the last three joint axes intersect at the wrist center point P as shown in Fig. 2.8. Hence rotations of the last three joints do not affect the position of P. Figure 2.9 shows the end-effector coordinate system (x_6, y_6, z_6), the wrist center P, and the vector relation between them.

The wrist center position with respect to and expressed in the end-effector coordinate system is

$$
{}^{6}\mathbf{p} = \overline{QP} = [0, 0, -d_6, 1]^{\mathrm{T}}. \tag{2.76}
$$

The wrist center position with respect to and expressed in the base coordinate system is

$$
{}^{0}\mathbf{p} = \overline{OP} = \begin{bmatrix} p_x \\ p_x \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} q_x - d_6 w_x \\ q_x - d_6 w_y \\ q_z - d_6 w_z \\ 1 \end{bmatrix}. \tag{2.77}
$$

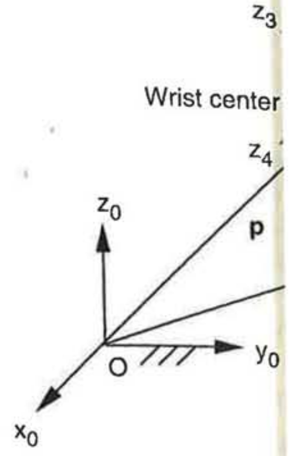

FIGURE 2.9. Hand c

Hence, given the end-effector
center point P with respect to
observe from Fig. 2.8 that the
the link 3 coordinate system is

${}^{3}\mathbf{p} =$

Transforming ${}^{3}\mathbf{p}$ into the b

Equation (2.79) consists of th
the position and orientation of

Theoretically, we can solv
follows we take a simpler app
inverse of ${}^{0}A_1$, we obtain

(

Substituting Eqs. (2.66) throu

$p_x c\theta_1 + p_y s\theta_1$

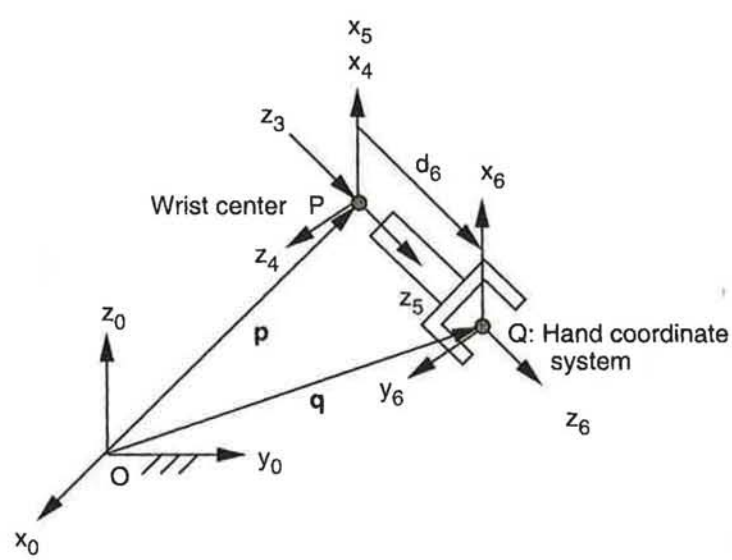

FIGURE 2.9. Hand coordinate system and wrist center position.

Hence, given the end-effector location, we can find the position of the wrist center point P with respect to the base coordinate system. Furthermore, we observe from Fig. 2.8 that the position of the wrist center P with respect to the link 3 coordinate system is given by

$$ {}^{3}\mathbf{p} = \overline{CP} = [0, 0, d_4, 1]^{\mathrm{T}}. \tag{2.78} $$

Transforming ${}^{3}\mathbf{p}$ into the base coordinate system, we obtain

$$ {}^{0}\mathbf{p} = {}^{0}A_3\, {}^{3}\mathbf{p}. \tag{2.79} $$

Equation (2.79) consists of three scalar equations in three unknowns. Hence the position and orientation of the inverse kinematics problem are decoupled.

Theoretically, we can solve Eq. (2.79) for the three joint angles. In what follows we take a simpler approach. Multiplying both sides Eq. (2.79) by the inverse of ${}^{0}A_1$, we obtain

$$ ({}^{0}A_1)^{-1}\, {}^{0}\mathbf{p} = {}^{1}A_3\, {}^{3}\mathbf{p}. \tag{2.80} $$

Substituting Eqs. (2.66) through (2.68) into (2.80) yields

$$ p_x c\theta_1 + p_y s\theta_1 - a_1 = a_2 c\theta_2 + a_3 c\theta_{23} + d_4 s\theta_{23}, \tag{2.81} $$

$$ p_z = a_2 s\theta_2 + a_3 s\theta_{23} - d_4 c\theta_{23}, \tag{2.82} $$

$$p_x s\theta_1 - p_y c\theta_1 = 0, \tag{2.83}$$

where p_x, p_y, and p_z are given by Eq. (2.77).

A solution for θ_1 is found immediately by solving Eq. (2.83).

$$\theta_1 = \tan^{-1} \frac{p_y}{p_x}. \tag{2.84}$$

Hence there are two solutions of θ_1. Specifically, if $\theta_1 = \theta_1^*$ is a solution, $\theta_1 = \theta_1^* + \pi$ is also a solution, where $\pi \geq \theta_1^* \geq 0$. We call $\theta_1 = \theta_1^*$ the front-reach solution and $\theta_1 = \theta_1^* + \pi$ the back-reach solution. Because of the four-bar linkage and other mechanical constraints, the back-reach solution is physically impossible.

An observation of the kinematic structure reveals that the distance between point A and the wrist center P is independent of θ_1 and θ_2, which implies that these two variables can be eliminated simultaneously. Summing the squares of Eqs. (2.81), (2.82), and (2.83), gives

$$\kappa_1 s\theta_3 + \kappa_2 c\theta_3 = \kappa_3, \tag{2.85}$$

where $\kappa_1 = 2a_2 d_4$, $\kappa_2 = 2a_2 a_3$, and $\kappa_3 = p_x^2 + p_y^2 + p_z^2 - 2p_x a_1 c\theta_1 - 2p_y a_1 s\theta_1 + a_1^2 - a_2^2 - a_3^2 - d_4^2$.

We can convert Eq. (2.85) into a polynomial by making use of the following trigonometric identities:

$$c\theta_3 = \frac{1 - t_3^2}{1 + t_3^2} \quad \text{and} \quad s\theta_3 = \frac{2t_3}{1 + t_3^2}, \quad \text{where} \quad t_3 = \tan\frac{\theta_3}{2}.$$

Substituting the trigonometric identities above into Eq. (2.85) yields

$$(\kappa_3 + \kappa_2)t_3^2 - 2\kappa_1 t_3 + (\kappa_3 - \kappa_2) = 0. \tag{2.86}$$

Hence

$$\frac{\theta_3}{2} = \tan^{-1} \frac{\kappa_1 \pm \sqrt{\kappa_1^2 + \kappa_2^2 - \kappa_3^2}}{\kappa_3 + \kappa_2}. \tag{2.87}$$

Equation (2.86) yields (1) two real roots if $\kappa_1^2 + \kappa_2^2 - \kappa_3^2 > 0$, (2) one double root if $\kappa_1^2 + \kappa_2^2 - \kappa_3^2 = 0$, and (3) no real roots if $\kappa_1^2 + \kappa_2^2 - \kappa_3^2 < 0$. When Eq. (2.86) yields a double root, the arm is either in a fully stretched or a folded-back configuration. On the other hand, if Eq. (2.86) yields no real roots, the position is not reachable. Figure 2.10 shows two different arm configurations, corresponding to the two solutions of θ_3.

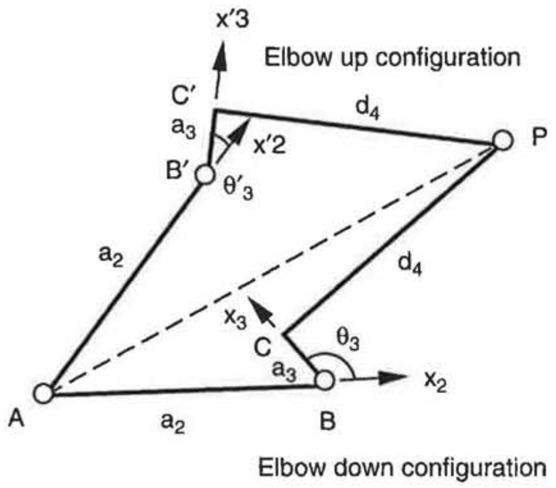

FIGURE 2.10. Two different arm configurations.

Once θ_1 and θ_3 are known, θ_2 can be obtained by back substitution. Expanding Eqs. (2.81) and (2.82), we obtain

$$\mu_1 c\theta_2 + \nu_1 s\theta_2 = \gamma_1, \tag{2.88}$$

$$\mu_2 c\theta_2 + \nu_2 s\theta_2 = \gamma_2, \tag{2.89}$$

where

$$\begin{aligned}
\mu_1 &= a_2 + a_3 c\theta_3 + d_4 s\theta_3,\\
\nu_1 &= -a_3 s\theta_3 + d_4 c\theta_3,\\
\gamma_1 &= p_x c\theta_1 + p_y s\theta_1 - a_1,\\
\mu_2 &= a_3 s\theta_3 - d_4 c\theta_3,\\
\nu_2 &= a_2 + a_3 c\theta_3 + d_4 s\theta_3,\\
\gamma_2 &= p_z.
\end{aligned}$$

Therefore, we can solve Eqs. (2.88) and (2.89) for $c\theta_2$ and $s\theta_2$. Once $s\theta_2$ and $c\theta_2$ are found, a unique value of θ_2 is obtained by taking

$$\theta_2 = \text{Atan2}(s\theta_2, c\theta_2). \tag{2.90}$$

We conclude that given the wrist center position, mathematically there are at most four possible arm configurations, but due to the mechanical limits, only two are physically possible.

(b) End-Effector Orientation. Once θ_1, θ_2, and θ_3 are solved, ${}^0A_3$ is completely known. The remaining joint angles can be found by multiplying both sides of Eq. (2.75) by $({}^0A_3)^{-1}$:

$$ {}^3A_6 = ({}^0A_3)^{-1}\,{}^0A_6. \tag{2.91} $$

We note that the elements on the right-hand side of Eq. (2.91) are known, and only the rotation part of Eq. (2.91) is needed for computation of the last three joint angles. The rotation matrices ${}^0R_3$ and ${}^3R_6$ are given by the upper 3×3 submatrices of Eqs. (2.73) and (2.74), respectively.

Equating the 3×3 element of Eq. (2.91) yields

$$ \theta_5 = \cos^{-1} r_{33}, \tag{2.92} $$

where $r_{33} = w_x c\theta_1 s\theta_{23} + w_y s\theta_1 s\theta_{23} - w_z c\theta_{23}$. Hence, corresponding to each solution set of θ_1, θ_2, and θ_3, Eq. (2.92) yields (1) two real roots if $|r_{33}| < 1$, and (2) $\theta_5 = 0$ or π if $|r_{33}| = 1$. When $\theta_5 = 0$ or π, the sixth joint axis, z_5, is in line with the fourth joint axis, z_3, and the wrist is said to be in a *singular configuration*. The condition $|r_{33}| > 1$ cannot physically arise.

Assuming that $s\theta_5 \neq 0$, we can solve θ_4 and θ_6 as follows. Equating the 1×3 element of Eq. (2.91) yields

$$ c\theta_4 = \frac{w_x c\theta_1 c\theta_{23} + w_y s\theta_1 c\theta_{23} + w_z s\theta_{23}}{s\theta_5}. \tag{2.93} $$

Equating the 2×3 element of Eq. (2.91) yields

$$ s\theta_4 = \frac{w_x s\theta_1 - w_y c\theta_1}{s\theta_5}. \tag{2.94} $$

Hence, corresponding to each solution set of θ_1, θ_2, θ_3, and θ_5, Eqs. (2.93) and (2.94) yield a unique solution of θ_4:

$$ \theta_4 = \mathrm{Atan2}(s\theta_4, c\theta_4). \tag{2.95} $$

Similarly, equating the 3×1 element of Eq. (2.91) yields

$$ c\theta_6 = -\frac{u_x c\theta_1 s\theta_{23} + u_y s\theta_1 s\theta_{23} - u_z c\theta_{23}}{s\theta_5}. \tag{2.96} $$

Equating the 3×2 element of Eq. (2.91) yields

$$ s\theta_6 = \frac{v_x c\theta_1 s\theta_{23} + v_y s\theta_1 s\theta_{23} - v_z c\theta_{23}}{s\theta_5}. \tag{2.97} $$

Hence, corresponding to each s
and (2.97) yield a unique soluti

θ_6 =

We conclude that correspon
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ble. However, due to mechanica
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2.6.4 Tsai and Morgan's

In this section we outline a so
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closed-form solutions for mani
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Figure 2.11 shows a genera
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tor coordinate system. Using th
equation can be written as

${}^0A_1\,{}^1A_2$

For convenience, we introduc
vector **e** of the z_5-axis of the
These two vectors can be exp
$[0, 0, 0, 1]^T$ and ${}^5\mathbf{e} = [0, 0, 1,$
${}^0\mathbf{p} = [p_x, p_y, p_z, 1]^T$ and $\mathbf{e} =$
and the vector **e** are attached
from the given end-effector lo

$$ \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = {}^0A_6\,{}^6A_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = $$

Hence, corresponding to each solution set of $\theta_1, \theta_2, \theta_3, \theta_4$, and θ_5, Eqs. (2.96) and (2.97) yield a unique solution of θ_6:

$$\theta_6 = \text{Atan2}(s\theta_6, c\theta_6). \tag{2.98}$$

We conclude that corresponding to each solution set of the first three joint angles, there are two possible wrist configurations. Since there are four possible upper arm configurations, a total of eight manipulator postures are possible. However, due to mechanical limits, fewer than eight manipulator postures are physically realizable. When $s\theta_5 = 0$, Eqs. (2.93) through (2.98) degenerate. For such a singular condition, only the sum or difference of θ_4 and θ_6 can be computed.

2.6.4 Tsai and Morgan's Solution

In this section we outline a solution method developed by Tsai and Morgan (1985), who reduced the problem to a system of four equations and then employed a numerical method, known as the *homotopy method*, to find all solutions to the inverse kinematics of a general $6R$ manipulator. They also derived closed-form solutions for manipulators in which three consecutive joint axes either intersect at a common point or are parallel to one another.

Figure 2.11 shows a general $6R$ manipulator where point Q denotes the origin and **u**, **v**, and **w** denote three orthogonal unit vectors of the end effector coordinate system. Using the Denavit–Hartenberg method, a loop-closure equation can be written as

$$^0A_1\, ^1A_2\, ^2A_3\, ^3A_4\, ^4A_5\, ^5A_6 = {}^0A_6. \tag{2.99}$$

For convenience, we introduce a position vector **p** of the origin and a unit vector **e** of the z_5-axis of the fifth coordinate system as shown in Fig. 2.11. These two vectors can be expressed in the fifth coordinate system as $^5\mathbf{p} = [0, 0, 0, 1]^T$ and $^5\mathbf{e} = [0, 0, 1, 0]^T$, or in the fixed coordinate system as $\mathbf{p} = {}^0\mathbf{p} = [p_x, p_y, p_z, 1]^T$ and $\mathbf{e} = {}^0\mathbf{e} = [e_x, e_y, e_z, 0]^T$. Since both the point P and the vector **e** are attached to the end effector, **p** and **e** can be computed from the given end-effector location as follows:

$$\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = {}^0A_6\, ^6A_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -u_x a_6 - (v_x s\alpha_6 + w_x c\alpha_6) d_6 + q_x \\ -u_y a_6 - (v_y s\alpha_6 + w_y c\alpha_6) d_6 + q_y \\ -u_z a_6 - (v_z s\alpha_6 + w_z c\alpha_6) d_6 + q_z \\ 1 \end{bmatrix}, \tag{2.100}$$

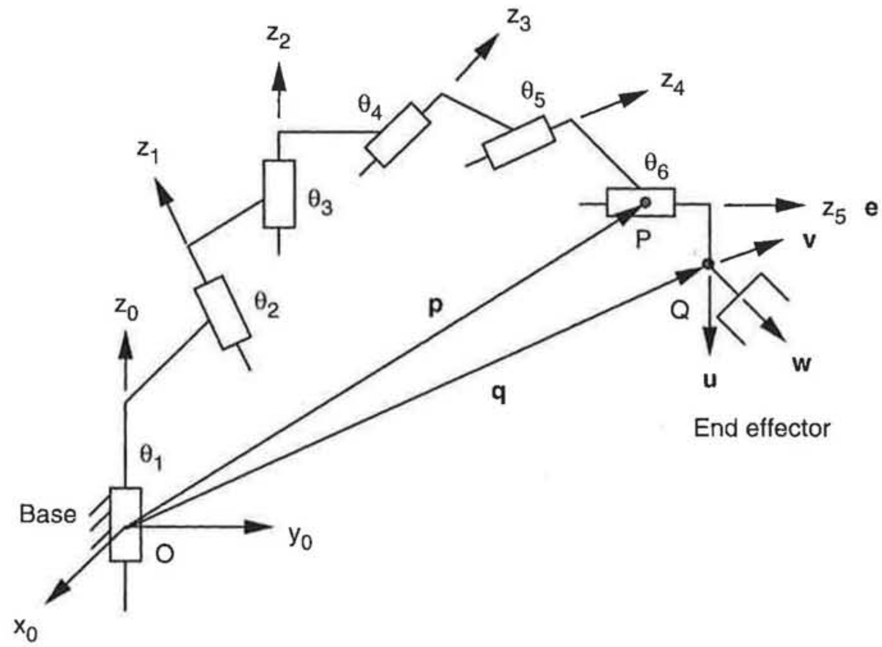

FIGURE 2.11. General 6*R* manipulator.

$$\begin{bmatrix} e_x \\ e_y \\ e_z \\ 0 \end{bmatrix} = {}^0A_6\,{}^6A_5 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} v_x \mathrm{s}\alpha_6 + w_x \mathrm{c}\alpha_6 \\ v_y \mathrm{s}\alpha_6 + w_y \mathrm{c}\alpha_6 \\ v_z \mathrm{s}\alpha_6 + w_z \mathrm{c}\alpha_6 \\ 0 \end{bmatrix}. \tag{2.101}$$

Equations (2.100) and (2.101) imply that once the end-effector location is given, the point P and the direction of z_5-axis can be found.

The transformation between ${}^5\mathbf{p}$ and ${}^0\mathbf{p}$ and between ${}^5\mathbf{e}$ and ${}^0\mathbf{e}$ can be written as

$${}^0\mathbf{p} = {}^0A_1\,{}^1A_2\,{}^2A_3\,{}^3A_4\,{}^4A_5\,{}^5\mathbf{p}, \tag{2.102}$$

$${}^0\mathbf{e} = {}^0A_1\,{}^1A_2\,{}^2A_3\,{}^3A_4\,{}^4A_5\,{}^5\mathbf{e}. \tag{2.103}$$

To simplify the analysis, we multiply both sides of Eqs. (2.102) and (2.103) by $({}^0A_1\,{}^1A_2)^{-1}$. The resulting equations can be written

$${}^2\mathbf{p} = {}^2\mathbf{p}', \tag{2.104}$$

$${}^2\mathbf{e} = {}^2\mathbf{e}', \tag{2.105}$$

where

$${}^2\mathbf{p} =$$
$${}^2\mathbf{p}' =$$
$${}^2\mathbf{e} =$$
$${}^2\mathbf{e}' =$$

are the position vectors of P an
the (x_2, y_2, z_2) coordinate syste
Equations (2.104) and (2.10
the variable θ_6. However, only t
are independent, because the c
a unit vector. Hence there are
knowns, $\theta_1, \ldots, \theta_5$. The x and
third-degree polynomials, whi
nomial in the sines and cosine
this approach, θ_6 does not app
reduces the complexity of the p
the system of equations above.

Elimination of θ_3. First, we n
and (2.105) are already free of
of Eqs. (2.104) and (2.105), yie

$$h_x \mathrm{s}\alpha_2 \mathrm{s}\theta_2 - h_y \mathrm{s}$$
$$= -h_z \mathrm{c}\alpha_2$$
$$n_x \mathrm{s}\alpha_2 \mathrm{s}\theta_2 - n_y \mathrm{s}$$
$$= -n_z \mathrm{c}\alpha_2$$

where

$$g_x = a_5 \mathrm{c}\theta_5 + a$$
$$g_y = -a_5 \mathrm{c}\alpha_4 \mathrm{s}\theta$$
$$g_z = a_5 \mathrm{s}\alpha_4 \mathrm{s}\theta_5 +$$
$$h_x = p_x \mathrm{c}\theta_1 + p$$
$$h_y = -p_x \mathrm{c}\alpha_1 \mathrm{s}\theta$$
$$h_z = p_x \mathrm{s}\alpha_1 \mathrm{s}\theta_1 +$$
$$m_x = \mathrm{c}\alpha_5 \mathrm{s}\theta_5,$$

where

$$
\begin{aligned}
{}^{2}\mathbf{p} &= {}^{2}A_3\,{}^{3}A_4\,{}^{4}A_5\,{}^{5}\mathbf{p},\\
{}^{2}\mathbf{p}' &= ({}^{1}A_2)^{-1}({}^{0}A_1)^{-1}\,{}^{0}\mathbf{p},\\
{}^{2}\mathbf{e} &= {}^{2}A_3\,{}^{3}A_4\,{}^{4}A_5\,{}^{5}\mathbf{e},\\
{}^{2}\mathbf{e}' &= ({}^{1}A_2)^{-1}\,({}^{0}A_1)^{-1}\,{}^{0}\mathbf{e}
\end{aligned}
$$

are the position vectors of P and the direction of the z_5-axis with reference to the (x_2, y_2, z_2) coordinate system.

Equations (2.104) and (2.105) constitute a set of six scalar equations free of the variable θ_6. However, only two of the three scalar equations in Eq. (2.105) are independent, because the components of $\mathbf{e}$ must satisfy the condition of a unit vector. Hence there are only five independent equations in five unknowns, $\theta_1, \ldots, \theta_5$. The x and y components in Eqs. (2.104) and (2.105) are third-degree polynomials, while the z-component is a second-degree polynomial in the sines and cosines of five joint angles. We note that by using this approach, θ_6 does not appear in the system of equations and therefore reduces the complexity of the problem. In the following we eliminate θ_3 from the system of equations above.

Elimination of θ_3. First, we notice that both z-components of Eqs. (2.104) and (2.105) are already free of the variable θ_3. Expanding the z-components of Eqs. (2.104) and (2.105), yields

$$
\begin{aligned}
&h_x \mathrm{s}\alpha_2 \mathrm{s}\theta_2 - h_y \mathrm{s}\alpha_2 \mathrm{c}\theta_2 - g_x \mathrm{s}\alpha_3 \mathrm{s}\theta_4 + g_y \mathrm{s}\alpha_3 \mathrm{c}\theta_4\\
&\qquad = -h_z \mathrm{c}\alpha_2 + d_2 \mathrm{c}\alpha_2 + g_z \mathrm{c}\alpha_3 + d_3,
\end{aligned}
\tag{2.106}
$$

$$
\begin{aligned}
&n_x \mathrm{s}\alpha_2 \mathrm{s}\theta_2 - n_y \mathrm{s}\alpha_2 \mathrm{c}\theta_2 - m_x \mathrm{s}\alpha_3 \mathrm{s}\theta_4 + m_y \mathrm{s}\alpha_3 \mathrm{c}\theta_4\\
&\qquad = -n_z \mathrm{c}\alpha_2 + m_z \mathrm{c}\alpha_3,
\end{aligned}
\tag{2.107}
$$

where

$$
\begin{aligned}
g_x &= a_5 \mathrm{c}\theta_5 + a_4,\\
g_y &= -a_5 \mathrm{c}\alpha_4 \mathrm{s}\theta_5 + d_5 \mathrm{s}\alpha_4,\\
g_z &= a_5 \mathrm{s}\alpha_4 \mathrm{s}\theta_5 + d_5 \mathrm{c}\alpha_4 + d_4,\\
h_x &= p_x \mathrm{c}\theta_1 + p_y \mathrm{s}\theta_1 - a_1,\\
h_y &= -p_x \mathrm{c}\alpha_1 \mathrm{s}\theta_1 + p_y \mathrm{c}\alpha_1 \mathrm{c}\theta_1 + (p_z - d_1)\mathrm{s}\alpha_1,\\
h_z &= p_x \mathrm{s}\alpha_1 \mathrm{s}\theta_1 - p_y \mathrm{s}\alpha_1 \mathrm{c}\theta_1 + (p_z - d_1)\mathrm{c}\alpha_1,\\
m_x &= \mathrm{c}\alpha_5 \mathrm{s}\theta_5,
\end{aligned}
$$

$$
\begin{aligned}
m_y &= c\alpha_4 s\alpha_5 c\theta_5 + s\alpha_4 c\alpha_5,\\
m_z &= -s\alpha_4 s\alpha_5 c\theta_5 + c\alpha_4 c\alpha_5,\\
n_x &= e_x c\theta_1 + e_y s\theta_1,\\
n_y &= -e_x c\alpha_1 s\theta_1 + e_y c\alpha_1 c\theta_1 + e_z s\alpha_1,\\
n_z &= e_x s\alpha_1 s\theta_1 - e_y s\alpha_1 c\theta_1 + e_z c\alpha_1.
\end{aligned}
$$

A third equation that is free of θ_3 is obtained by performing the dot product ${}^2\mathbf{p} \cdot {}^2\mathbf{e} = {}^2\mathbf{p}' \cdot {}^2\mathbf{e}'$. Substituting Eqs. (2.104) and (2.105) into the dot product and simplifying, we obtain

$$
\begin{aligned}
&a_2 n_y s\theta_2 + a_2 n_x c\theta_2 + (a_3 m_y + d_3 m_x s\alpha_3) s\theta_4 + (a_3 m_x - d_3 m_y s\alpha_3) c\theta_4\\
&\quad = -a_1 n_x - d_2 n_z - a_4 m_x - m_z (d_3 c\alpha_3 + d_4) + k_1,
\end{aligned}
\tag{2.108}
$$

where $k_1 = -d_5 c\alpha_5 + p_x e_x + p_y e_y + (p_z - d_1) e_z$.

A fourth equation that is free of θ_3 is obtained by equating the sum of the squares of the x, y, and z components on both sides of Eq. (2.104). Expanding $({}^2\mathbf{p})^2 = ({}^2\mathbf{p}')^2$ yields

$$
\begin{aligned}
&a_2 h_y s\theta_2 + a_2 h_x c\theta_2 + (a_3 g_y + d_3 g_x s\alpha_3) s\theta_4 + (a_3 g_x - d_3 g_y s\alpha_3) c\theta_4\\
&\quad = -a_1 h_x - d_2 h_z - a_4 g_x - g_z (d_3 c\alpha_3 + d_4) + k_2,
\end{aligned}
\tag{2.109}
$$

where

$$
k_2 = 0.5[p_x^2 + p_y^2 + (p_z - d_1)^2 - a_1^2 + a_2^2 + d_2^2 - a_3^2 - d_3^2 + a_4^2 + d_4^2 - a_5^2 - d_5^2].
$$

Equations (2.106) through (2.109) represent a system of four second-degree polynomials in the sines and cosines of four joint angles. We may consider $\sin\theta_i$ and $\cos\theta_i$ as two independent variables and add the following trigonometric identities as supplementary equations of constraint:

$$
s^2\theta_i + c^2\theta_i = 1, \qquad \text{for} \quad i = 1, 2, 4, 5. \tag{2.110}
$$

In this way, we obtain a system of eight second-degree polynomials in eight variables. Tsai and Morgan employed a continuation method to solve the system of equations above and showed that the most general 6-dof, $6R$ robot has at most 16 significant solutions. See Appendix A for more details.

The system of equations will decouple when any three consecutive joint axes either intersect at a common point or are parallel to one another. For these special geometries, closed-form solutions can be derived. In what follows we illustrate the decoupling by solving the inverse kinematics of two special cases that are most commonly implemented in industrial robots. Other special cases can be derived by applying the kinematic inversions.

$+ e_z s\alpha_1,$

$- e_z c\alpha_1.$

ned by performing the dot product
and (2.105) into the dot product

$)s\theta_4 + (a_3 m_x - d_3 m_y s\alpha_3)c\theta_4$

$+ d_4) + k_1,$ (2.108)

$- d_1)e_z.$

ained by equating the sum of the
h sides of Eq. (2.104). Expanding

$s\theta_4 + (a_3 g_x - d_3 g_y s\alpha_3)c\theta_4$

$+ d_4) + k_2,$ (2.109)

$- a_3^2 - d_3^2 + a_4^2 + d_4^2 - a_5^2 - d_5^2].$

sent a system of four second-
s of four joint angles. We may
variables and add the following
uations of constraint:

$i = 1, 2, 4, 5.$ (2.110)

ond-degree polynomials in eight
nuation method to solve the sys-
nost general 6-dof, 6R robot has
x A for more details.
hen any three consecutive joint
are parallel to one another. For
ns can be derived. In what fol-
the inverse kinematics of two
ented in industrial robots. Other
kinematic inversions.

(a) Last Three Joint Axes Intersecting at a Common Point. When the last three joint axes intersect at a common point, $a_4 = a_5 = d_5 = 0$ identically. Substituting these values into Eqs. (2.106) and (2.109) yields

$$h_x s\theta_2 - h_y c\theta_2 = \mu_1 / s\alpha_2, \tag{2.111}$$

$$h_y s\theta_2 + h_x c\theta_2 = \mu_2 / a_2, \tag{2.112}$$

provided that $s\alpha_2 \neq 0$ and $a_2 \neq 0$, where

$$\mu_1 = -h_z c\alpha_2 + d_2 c\alpha_2 + d_3 + d_4 c\alpha_3,$$

$$\mu_2 = -a_1 h_x - d_2 h_z - d_3 d_4 c\alpha_3 + 0.5[p_x^2 + p_y^2 + (p_z - d_1)^2 - a_1^2 + a_2^2 + d_2^2 - a_3^2 - d_3^2 - d_4^2].$$

Equations (2.111) and (2.112) contain only two unknown variables and hence are completely decoupled from Eqs. (2.107) and (2.108). To eliminate θ_2, we sum the squares of Eqs. (2.111) and (2.112).

$$h_x^2 + h_y^2 = (\mu_1 / s\alpha_2)^2 + (\mu_2 / a_2)^2. \tag{2.113}$$

Equation (2.113) contains only one variable, θ_1. We may convert it into a fourth-degree polynomial in t_1 by replacing $s\theta_1$ with $2t_1/(1 + t_1^2)$ and $c\theta_1$ with $(1 - t_1^2)/(1 + t_1^2)$, where $t_1 = \tan(\theta_1/2)$. Hence, for each given end-effector position, there are at most four real solutions of θ_1. Once θ_1 is found, a unique solution of θ_2 can be obtained by solving Eqs. (2.111) and (2.112) simultaneously for $s\theta_2$ and $c\theta_2$, and then applying the two-argument arctangent function. Following that, a unique solution of θ_3 can be found by solving the two scalar equations associated with the x and y components of Eq. (2.104).

Corresponding to each solution set of $(\theta_1, \theta_2, \theta_3)$, two sets of $(\theta_4, \theta_5, \theta_6)$ can be found by following the procedure outlined in the earlier example. Hence we conclude that there are at most eight possible solutions sets (manipulator postures).

(b) Joint Axes 2, 3, and 4 Parallel to One Another. When the second, third, and fourth joint axes are parallel to one another, $\alpha_2 = \alpha_3 = 0$ identically. Further, since the common normals between the second and third joint axes and between the third and fourth joint axes are indeterminate, we can always define these two common normals such that $d_2 = d_3 = 0$. With these values, Eqs. (2.106) and (2.107) reduce to

$$s\alpha_4 s\theta_5 = \frac{h_z - d_5 c\alpha_4 - d_4}{a_5}, \tag{2.114}$$

$$s\alpha_4 c\theta_5 = \frac{-n_z + c\alpha_4 c\alpha_5}{s\alpha_5}, \tag{2.115}$$

provided that $a_5 \neq 0$ and $s\alpha_5 \neq 0$.

Again, Eqs. (2.114) and (2.115) contain only two unknown variables and hence are completely decoupled from Eqs. (2.108) and (2.109). We can eliminate θ_5 by summing the squares of Eq. (2.114) and (2.115):

$$s^2\alpha_4 = \left(\frac{h_z - d_5 c\alpha_4 - d_4}{a_5}\right)^2 + \left(\frac{-n_z + c\alpha_4 c\alpha_5}{s\alpha_5}\right)^2. \tag{2.116}$$

Equation (2.116) contains only one variable, θ_1. We may convert it into a fourth-degree polynomial in t_1 by replacing $s\theta_1$ with $2t_1/(1+t_1^2)$ and $c\theta_1$ with $(1-t_1^2)/(1+t_1^2)$, where $t_1 = \tan(\theta_1/2)$. Hence for each given end-effector position and orientation, there are at most four real solutions of θ_1. Once θ_1 is known, a unique solution of θ_5 can be obtained by solving Eqs. (2.114) and (2.115) simultaneously for $s\theta_5$ and $c\theta_5$ and then applying the two-argument arctangent function.

When $\alpha_2 = \alpha_3 = 0$, the two scalar equations corresponding to the x and y components of Eq. (2.105) reduce to

$$m_x c\theta_{34} + m_y s\theta_{34} = n_x c\theta_2 + n_y s\theta_2, \tag{2.117}$$

$$m_x s\theta_{34} - m_y c\theta_{34} = -n_x s\theta_2 + n_y c\theta_2. \tag{2.118}$$

Equations (2.117) and (2.118) contain two unknown variables, θ_{34} and θ_2. We may reduce these two equations to a single equation in one variable by the following procedure. Subtracting Eq. (2.118) $\times$ $s\theta_2$ from (2.117) $\times$ $c\theta_2$ yields

$$m_x c\theta_{234} + m_y s\theta_{234} = n_x. \tag{2.119}$$

Adding Eq. (2.118) $\times$ $c\theta_2$ to (2.117) $\times$ $s\theta_2$ yields

$$m_x s\theta_{234} - m_y c\theta_{234} = n_y. \tag{2.120}$$

Hence, corresponding to each solution set of θ_1 and θ_5, a unique solution of θ_{234} can be obtained by solving Eqs. (2.119) and (2.120) for $s\theta_{234}$ and $c\theta_{234}$, and then applying the two-argument arctangent function.

Similarly, the two scalar equations corresponding to the x and y components of Eq. (2.104) reduce to

$$g_x c\theta_{34} + g_y s\theta_{34} + a_3 c\theta_3 = h_x c\theta_2 + h_y s\theta_2 - a_2, \tag{2.121}$$

$$g_x s\theta_{34} - g_y c\theta_{34} + a_3 s\theta_3 = -h_x s\theta_2 + h_y c\theta_2. \tag{2.122}$$

Subtracting Eq. (2.122) $\times$ $s\theta_2$ from (2.121) $\times$ $c\theta_2$ yields

$$g_x c\theta_{234} + g_y s\theta_{234} + a_3 c\theta_{23} = h_x - a_2 c\theta_2. \tag{2.123}$$

Adding Eq. (2.122) $\times$ $c\theta_2$ to (2.121) $\times$ $s\theta_2$ yields

$$g_x s\theta_{234} - g_y c\theta_{234} + a_3 s\theta_{23} = h_y - a_2 s\theta_2. \tag{2.124}$$

Summing the squares of [Eq. (2.123) - h_x] and [Eq. (2.124) - h_y] yields

$$2a_3(k_3 c\theta_{23} + k_4 s\theta_{23}) + k_5 = 0, \tag{2.125}$$

where

$$k_3 = g_x c\theta_{234} + g_y s\theta_{234} - h_x,$$
$$k_4 = g_x s\theta_{234} - g_y c\theta_{234} - h_y,$$
$$k_5 = -a_2^2 + a_3^2 + k_3^2 + k_4^2.$$

Hence, corresponding to each solution set of $(\theta_1, \theta_5, \theta_{234})$, Eq. (2.125) yields two solutions of θ_{23}. Once θ_{23} is found, Eqs. (2.123) and (2.124) yield a unique solution of θ_2. We conclude that there are at most eight possible solution sets.

2.7 METHOD OF SUCCESSIVE SCREW DISPLACEMENTS

In this section we study a method of analysis based on the concept of *successive screw displacements*. First, the transformation matrix associated with a screw displacement is derived. Then the concept of the *resultant screw* of two successive screw displacements is described. Then the concept is applied to the position analysis of serial manipulators.

2.7.1 Transformation Based on Screw Displacement

Chasles' theorem states that the general spatial displacement of a rigid body is a translation plus a rotation. A stronger form of the theorem states that regardless of how a rigid body is displaced from one location to another, the displacement can be regarded as a rotation about and a translation along some axis. Such a combination of translation and rotation is called a *screw displacement* (Bottema and Roth, 1979). In what follows we derive a homogeneous transformation based on the concept of screw displacement.

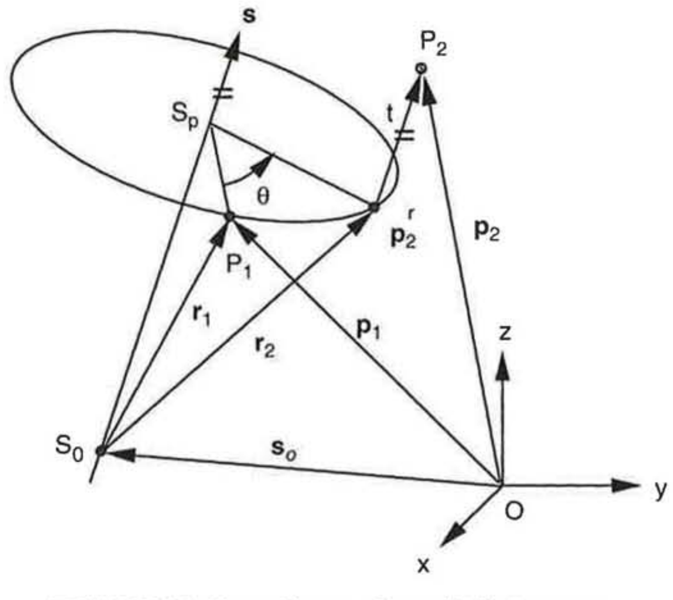

FIGURE 2.12. Vector diagram of a spatial displacement.

Figure 2.12 shows a point P that is displaced from a first position P_1 to a second position P_2 by a rotation of θ about a screw axis followed by a translation of t along the same axis. The rotation brings P from P_1 to P_2^r, and the translation brings P from P_2^r to P_2. In the figure, $\mathbf{s} = [s_x, s_y, s_z]^{\mathrm{T}}$ denotes a unit vector along the direction of the screw axis, and $\mathbf{s}_o = [s_{ox}, s_{oy}, s_{oz}]^{\mathrm{T}}$ denotes the position vector of a point lying on the screw axis. The rotation angle θ and the translational distance t are called the *screw parameters*. The screw axis together with the screw parameters completely define the general displacement of a rigid body. Note that for a general displacement of a rigid body, the screw axis does not necessarily pass through the origin of the fixed frame.

The displacement equation due to a rotation about an axis passing through the origin was derived in Chapter 1. Hence we only need to take care of the fact that the screw axis does not pass through the origin and add the contribution due to a translation along the screw axis. Referring to Fig. 2.12, we observe that

$$\mathbf{r}_1 = \mathbf{p}_1 - \mathbf{s}_o, \tag{2.126}$$

$$\mathbf{r}_2 = \mathbf{p}_2 - \mathbf{s}_o - t\mathbf{s}. \tag{2.127}$$

Substituting Eqs. (2.126) and (2.1

$\mathbf{p}_2 = \mathbf{s}_o + t\mathbf{s} + (\mathbf{p}_1 - \mathbf{s}_o)c\theta + \mathbf{s}$

Equation (2.128) is known as R
displacement of a rigid body. Exp
and $\mathbf{p}_2$ by ${}^A\mathbf{p}$, we obtain

${}^A\mathbf{p} =$

where the elements of the rotatio
the position of the origin, ${}^A\mathbf{q}$, of th

$q_x = ts_x - s_{ox}(a_1$
$q_y = ts_y - s_{ox}a_{21}$
$q_z = ts_z - s_{ox}a_{31}$

Equation (2.129) can be written

where A is a 4×4 transformation

$a_{11} = (s_x^2 - 1)(1$
$a_{12} = s_x s_y(1 - c\theta$
$a_{13} = s_x s_z(1 - c\theta$
$a_{21} = s_y s_x(1 - c\theta$
$a_{22} = (s_y^2 - 1)(1$
$a_{23} = s_y s_z(1 - c\theta$
$a_{31} = s_z s_x(1 - c\theta$
$a_{32} = s_z s_y(1 - c\theta$
$a_{33} = (s_z^2 - 1)(1$
$a_{14} = ts_x - s_{ox}(a$
$a_{24} = ts_y - s_{ox}a_{2}$
$a_{34} = ts_z - s_{ox}a_{3}$

Substituting Eqs. (2.126) and (2.127) into (1.35), we obtain

$$\mathbf{p}_2 = \mathbf{s}_o + t\mathbf{s} + (\mathbf{p}_1 - \mathbf{s}_o)c\theta + \mathbf{s} \times (\mathbf{p}_1 - \mathbf{s}_o)s\theta + [(\mathbf{p}_1 - \mathbf{s}_o)^{\mathrm{T}}\mathbf{s}]\mathbf{s}(1 - c\theta). \tag{2.128}$$

Equation (2.128) is known as *Rodrigues's formula* for the general spatial displacement of a rigid body. Expanding Eq. (2.128) and replacing $\mathbf{p}_1$ by ${}^B\mathbf{p}$ and $\mathbf{p}_2$ by ${}^A\mathbf{p}$, we obtain

$${}^A\mathbf{p} = {}^AR_B\,{}^B\mathbf{p} + {}^A\mathbf{q}, \tag{2.129}$$

where the elements of the rotation matrix, a_{ij}, are given by Eq. (1.37), and the position of the origin, ${}^A\mathbf{q}$, of the moving frame is given by

$$\begin{aligned}
q_x &= ts_x - s_{ox}(a_{11} - 1) - s_{oy}a_{12} - s_{oz}a_{13},\\
q_y &= ts_y - s_{ox}a_{21} - s_{oy}(a_{22} - 1) - s_{oz}a_{23},\\
q_z &= ts_z - s_{ox}a_{31} - s_{oy}a_{32} - s_{oz}(a_{33} - 1).
\end{aligned} \tag{2.130}$$

Equation (2.129) can be written as a homogeneous transformation:

$${}^A\hat{\mathbf{p}} = A\,{}^B\hat{\mathbf{p}} \tag{2.131}$$

where A is a 4×4 transformation matrix the elements of which are given by

$$\begin{aligned}
a_{11} &= (s_x^2 - 1)(1 - c\theta) + 1,\\
a_{12} &= s_x s_y(1 - c\theta) - s_z s\theta,\\
a_{13} &= s_x s_z(1 - c\theta) + s_y s\theta,\\
a_{21} &= s_y s_x(1 - c\theta) + s_z s\theta,\\
a_{22} &= (s_y^2 - 1)(1 - c\theta) + 1,\\
a_{23} &= s_y s_z(1 - c\theta) - s_x s\theta,\\
a_{31} &= s_z s_x(1 - c\theta) - s_y s\theta,\\
a_{32} &= s_z s_y(1 - c\theta) + s_x s\theta,\\
a_{33} &= (s_z^2 - 1)(1 - c\theta) + 1,\\
a_{14} &= ts_x - s_{ox}(a_{11} - 1) - s_{oy}a_{12} - s_{oz}a_{13},\\
a_{24} &= ts_y - s_{ox}a_{21} - s_{oy}(a_{22} - 1) - s_{oz}a_{23},\\
a_{34} &= ts_z - s_{ox}a_{31} - s_{oy}a_{32} - s_{oz}(a_{33} - 1),
\end{aligned}$$

$$
\begin{aligned}
a_{41} &= 0,\\
a_{42} &= 0,\\
a_{43} &= 0,\\
a_{44} &= 1.
\end{aligned}
\tag{2.132}
$$

The upper left 3×3 submatrix of A represents the rotation of the rigid body. The upper right 3×1 submatrix represents the translation of the origin Q (i.e., $a_{14} = q_x, a_{24} = q_y$, and $a_{34} = q_z$). This representation of a spatial displacement requires eight parameters: three associated with the direction of the screw axis, three associated with the location of the screw axis, one associated with the rotation angle, and one associated with the translational distance. However, only two of the three parameters associated with the direction of the screw axis are independent since they must satisfy the condition of a unit vector:

$$\mathbf{s}^{\mathrm{T}}\mathbf{s} = 1. \tag{2.133}$$

Similarly, only two of the three parameters associated with the location of the screw axis are independent, since S_o can be any point on the screw axis. For convenience, we may choose $\mathbf{s}_o$ to be normal to the screw axis:

$$\mathbf{s}_o^{\mathrm{T}}\mathbf{s} = 0. \tag{2.134}$$

Given the screw axis and screw parameters, we can compute the elements of the transformation matrix by Eq. (2.132). On the other hand, given the spatial displacement of a rigid body in terms of a rotation matrix, ${}^AR_B$, and a translation vector, ${}^A\mathbf{q}$, we can compute the screw axis and the screw parameters as follows. The angle of rotation is given by

$$\theta = \cos^{-1}\frac{a_{11} + a_{22} + a_{33} - 1}{2}. \tag{2.135}$$

There are two solutions of θ, one being the negative of the other. Once the rotation angle is known, the direction of the screw axis is computed by

$$
\begin{aligned}
s_x &= \frac{a_{32} - a_{23}}{2s\theta},\\
s_y &= \frac{a_{13} - a_{31}}{2s\theta},\\
s_z &= \frac{a_{21} - a_{12}}{2s\theta}.
\end{aligned}
\tag{2.136}
$$

The translational distance i

and the screw axis location
tions in Eq. (2.130) along
there exists one solution co

From the derivation abc
screw axis, one being the n
represent the same screw, si
the $-(\mathbf{s}, \mathbf{s}_o)$ screw axis proc
$+t$ translation along the +(

2.7.2 Successive Scre

We now apply the concept
open-loop chains. Figure 2
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and $\$_2$, respectively. The fi
to the fixed base, and the se

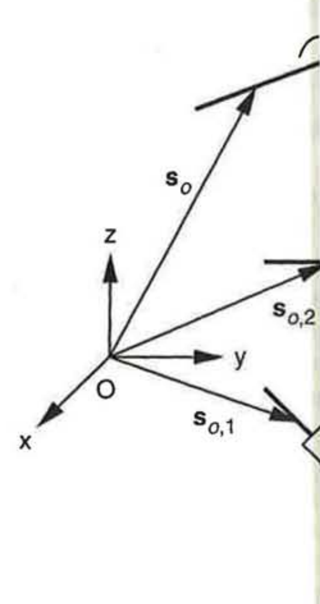

FIGURE 2.13. Two-li

successive screw displacements: that is, a rotation about the nth joint axis, followed by another about the $(n-1)$th joint axis, and so on. Since all screw displacements take place about the joint axes at the reference position, the resulting screw displacement is obtained by premultiplying these screw displacements:

$$A_h = A_1 A_2 \cdots A_{n-1} A_n. \tag{2.139}$$

Using the method of successive screw displacements, only one fixed coordinate system and one end effector coordinate system are needed. The screw parameters used in Eq. (2.132) should not be confused with the Denavit–Hartenberg parameters. The joint variables of a screw displacement represent the actual angles of rotation and/or distances of translation needed to bring the end effector from a reference position to a target position. Specifically, for a revolute joint, θ_i is a variable and $t_i = 0$ identically, while for a prismatic joint, t_i is a variable and $\theta_i = 0$ identically.

The D-H parameters do not represent the angle of rotation or the distance of translation about a joint axis. To obtain the actual displacements, it is necessary to subtract the joint variables associated with a reference position from that of a target position. One of the advantages of using successive screw displacements is that the reference position can be chosen arbitrarily. For example, it can be chosen at the *home position* of a robot, where all the information regarding the location of the end effector and the locations of the joint axes are known.

For direct kinematics, we compute Eq. (2.139) directly by using the given joint variables. For inverse kinematics, the left-hand side of Eq. (2.139) is given and the problem is to find the joint displacements needed to bring the hand to a desired location.

2.7.3 Position Analysis of an Elbow Manipulator

Figure 2.15 shows the schematic diagram of an elbow manipulator. In this manipulator, the second joint axis intersects the first perpendicularly, the third and fourth joint axes are parallel to the second, the fifth joint axis is perpendicular to the fourth with a small offset distance a_4, and the sixth joint axis intersects the fifth perpendicularly. We wish to solve the inverse kinematics problem of this manipulator using the method of successive screw displacements.

(a) Reference Position. First we identify a reference configuration with respect to which the displacement of the manipulator will be measured. Figure 2.16 shows such a reference configuration, where the first joint axis, $\$_1$,

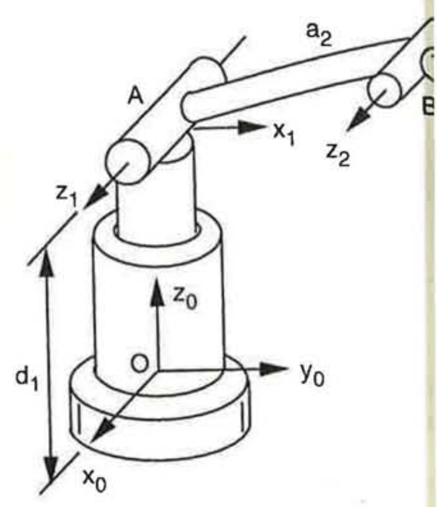

FIGURE 2.15. A

points up vertically in the positive
joint axes, $\$_2$, $\$_3$, and $\$_4$, are all poi
$\$_5$, points in the positive z-directior
positive x-direction. The hand coo
that the w_0-axis points in the positiv

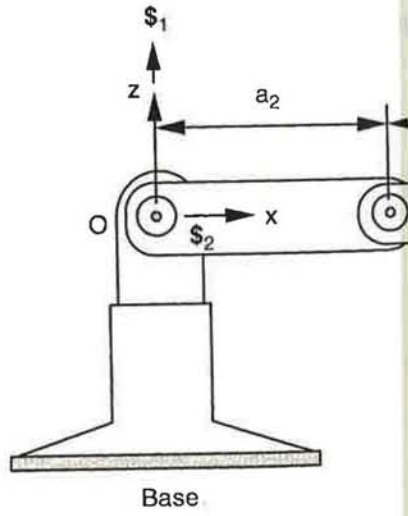

FIGURE 2.16. Referenc

rotation about the nth joint axis,
nt axis, and so on. Since all screw
xes at the reference position, the
y premultiplying these screw dis-

$$_{n-1}A_n. \qquad (2.139)$$

splacements, only one fixed coor-
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target position. Specifically, for a
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e angle of rotation or the distance
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an be chosen arbitrarily. For ex-
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ector and the locations of the joint

2.139) directly by using the given
left-hand side of Eq. (2.139) is
isplacements needed to bring the

Manipulator

of an elbow manipulator. In this
the first perpendicularly, the third
ond, the fifth joint axis is perpen-
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anipulator will be measured. Fig-
on, where the first joint axis, $\$_1$,

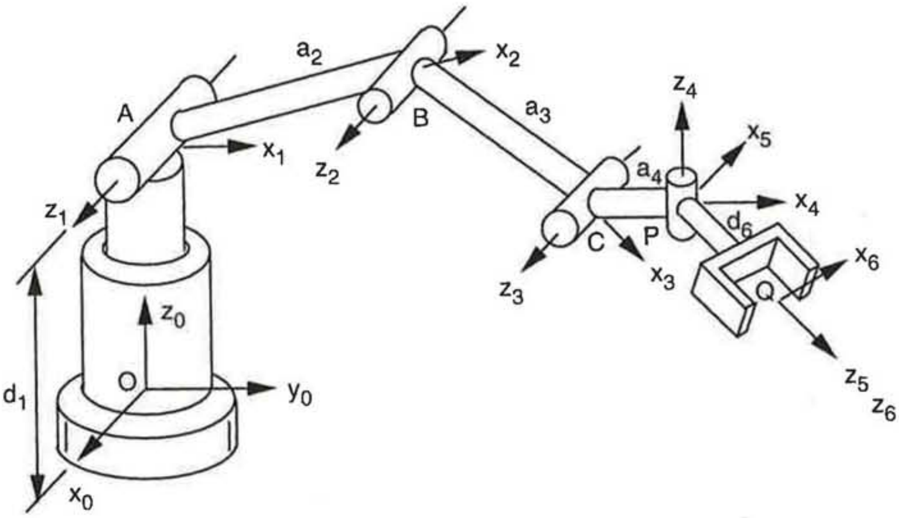

FIGURE 2.15. A 6-dof elbow manipulator.

points up vertically in the positive z-direction; the second, third, and fourth joint axes, $\$_2$, $\$_3$, and $\$_4$, are all pointing out of the paper; the fifth joint axis, $\$_5$, points in the positive z-direction; and the sixth joint axis, $\$_6$, points in the positive x-direction. The hand coordinate system is located at point Q such that the w_0-axis points in the positive x-direction and the u_0-axis points in the

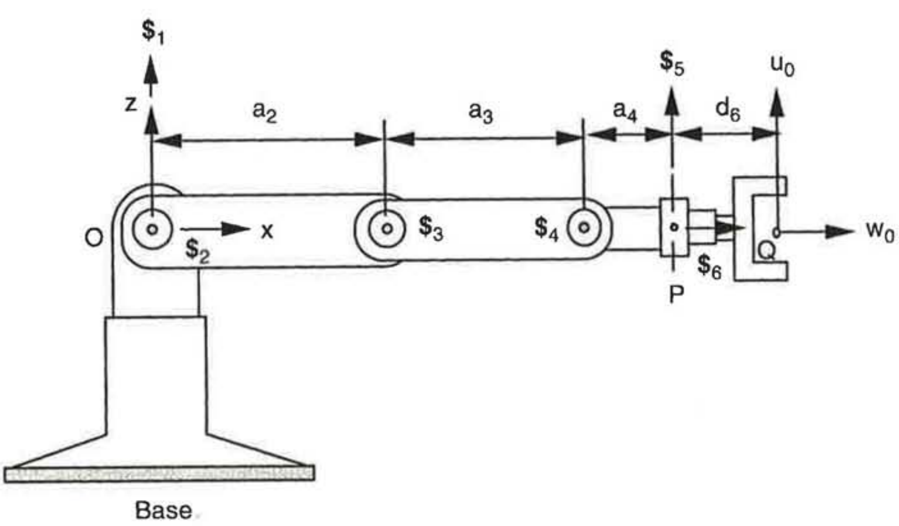

FIGURE 2.16. Reference position of the elbow manipulator.

TABLE 2.5. Screw Axis Locations of the Elbow Manipulator

Joint i	$\mathbf{s}_i$	$\mathbf{s}_{oi}$
1	$(0, 0, 1)$	$(0, 0, 0)$
2	$(0, -1, 0)$	$(0, 0, 0)$
3	$(0, -1, 0)$	$(a_2, 0, 0)$
4	$(0, -1, 0)$	$(a_2 + a_3, 0, 0)$
5	$(0, 0, 1)$	$(a_2 + a_3 + a_4, 0, 0)$
6	$(1, 0, 0)$	$(0, 0, 0)$

positive z-direction. At this reference position, the locations of the screw axes with respect to the fixed reference frame are listed in Table 2.5. The reference position of the end effector is

$$\mathbf{u}_0 = [0, 0, 1]^{\mathrm{T}}, \quad \mathbf{v}_0 = [0, -1, 0]^{\mathrm{T}}, \quad \mathbf{w}_0 = [1, 0, 0]^{\mathrm{T}}, \quad \text{and}$$
$$\mathbf{p}_0 = [a_2 + a_3 + a_4, 0, 0]^{\mathrm{T}}.$$

(b) Target Position. Let the target position of the end effector be

$$\mathbf{u} = [u_x, u_y, u_z]^{\mathrm{T}}, \quad \mathbf{v} = [v_x, v_y, v_z]^{\mathrm{T}}, \quad \mathbf{w} = [w_x, w_y, w_z]^{\mathrm{T}}, \quad \text{and}$$
$$\mathbf{p} = [p_x, p_y, p_z]^{\mathrm{T}}.$$

(c) Transformation Matrices. Substituting the coordinates of the joint axes into Eq. (2.132), we obtain the screw transformation matrices:

$$A_1 = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad ({}^0A_1)^{-1} = \begin{bmatrix} c\theta_1 & s\theta_1 & 0 & 0 \\ -s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} c\theta_2 & 0 & -s\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ s\theta_2 & 0 & c\theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} c\theta_3 & 0 & -s\theta_3 & a_2(1 - c\theta_3) \\ 0 & 1 & 0 & 0 \\ s\theta_3 & 0 & c\theta_3 & -a_2 s\theta_3 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} c\theta_4 & 0 & -s\theta_4 & (a_2+a_3)(1-c\theta_4) \\ 0 & 1 & 0 & 0 \\ s\theta_4 & 0 & c\theta_4 & -(a_2+a_3)s\theta_4 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & (a_2+a_3+a_4)(1-c\theta_5) \\ s\theta_5 & c\theta_5 & 0 & -(a_2+a_3+a_4)s\theta_5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\theta_6 & -s\theta_6 & 0 \\ 0 & s\theta_6 & c\theta_6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix products $A_2A_3A_4$ and $A_1A_2A_3A_4$ are computed as

$$A_2A_3A_4 = \begin{bmatrix} c\theta_{234} & 0 & -s\theta_{234} & a_2c\theta_2 + a_3c\theta_{23} - (a_2+a_3)c\theta_{234} \\ 0 & 1 & 0 & 0 \\ s\theta_{234} & 0 & c\theta_{234} & a_2s\theta_2 + a_3s\theta_{23} - (a_2+a_3)s\theta_{234} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.140}$$

$$A_1A_2A_3A_4 = \begin{bmatrix} c\theta_1c\theta_{234} & -s\theta_1 & -c\theta_1s\theta_{234} & c\theta_1[a_2c\theta_2 + a_3c\theta_{23} - (a_2+a_3)c\theta_{234}] \\ s\theta_1c\theta_{234} & c\theta_1 & -s\theta_1s\theta_{234} & s\theta_1[a_2c\theta_2 + a_3c\theta_{23} - (a_2+a_3)c\theta_{234}] \\ s\theta_{234} & 0 & c\theta_{234} & [a_2s\theta_2 + a_3s\theta_{23} - (a_2+a_3)s\theta_{234}] \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.141}$$

(d) Inverse Kinematics. The transformation of the wrist center point P is given by

$$\mathbf{p} = A_1A_2A_3A_4\mathbf{p}_0. \tag{2.142}$$

Multiplying both sides of the equation above by A_1^{-1}, we obtain

$$A_1^{-1}\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = A_2A_3A_4\begin{bmatrix} a_2+a_3+a_4 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{2.143}$$

Substituting A_1^{-1} and Eq. (2.140) into (2.143) yields

$$p_x c\theta_1 + p_y s\theta_1 = a_2 c\theta_2 + a_3 c\theta_{23} + a_4 c\theta_{234}, \tag{2.144}$$

$$-p_x s\theta_1 + p_y c\theta_1 = 0, \tag{2.145}$$

$$p_z = a_2 s\theta_2 + a_3 s\theta_{23} + a_4 s\theta_{234}. \tag{2.146}$$

From Eq. (2.145), two solutions of θ_1 are found immediately:

$$\theta_1 = \tan^{-1}\frac{p_y}{p_x}. \tag{2.147}$$

For this manipulator, the position and orientation are not decoupled. Therefore, we need to work on both simultaneously. Applying the transformation matrix to the approach vector $\mathbf{w}$ gives

$$R_1^{\mathrm{T}}\mathbf{w} = R_2 R_3 R_4 R_5 \mathbf{w}_0, \tag{2.148}$$

where R_i denotes the upper left 3×3 submatrix of A_i. Expanding Eq. (2.148), we obtain

$$w_x c\theta_1 + w_y s\theta_1 = c\theta_{234} c\theta_5, \tag{2.149}$$

$$-w_x s\theta_1 + w_y c\theta_1 = s\theta_5, \tag{2.150}$$

$$w_z = s\theta_{234} c\theta_5. \tag{2.151}$$

Corresponding to each solution of θ_1, Eq. (2.150) yields two solutions of θ_5:

$$\theta_5 = \sin^{-1}(-w_x s\theta_1 + w_y c\theta_1). \tag{2.152}$$

That is, if $\theta_5 = \theta_5^*$ is a solution, $\theta_5 = \pi - \theta_5^*$ is also a solution. Once θ_1 and θ_5 are known, Eqs. (2.149) and (2.151) can be solved for $s\theta_{234}$ and $c\theta_{234}$. This leads to a unique solution for θ_{234}:

$$\theta_{234} = \mathrm{Atan2}\left[w_z/c\theta_5, (w_x c\theta_1 + w_y s\theta_1)/c\theta_5\right]. \tag{2.153}$$

Next, we solve Eqs. (2.144) and (2.146) for θ_2 and θ_3. For convenience, we rewrite Eqs. (2.144) and (2.146) as follows:

$$a_2 c\theta_2 + a_3 c\theta_{23} = k_1, \tag{2.154}$$

$$a_2 s\theta_2 + a_3 s\theta_{23} = k_2, \tag{2.155}$$

where $k_1 = p_x c\theta_1 + p_y s\theta_1$
squares of Eqs. (2.154) and (

$a_2^2 +$

Hence

$\theta_3 =$

Therefore, corresponding to
most two real solutions of θ
a solution. Once θ_3 is know
and (2.155) simultaneously f
Finally, the solutions of θ_4 ar
To solve for θ_6, we apply

$(R_1$

Expanding Eq. (2.158), we c

$u_x c\theta_1 c\theta_{234} +$

$-u_x c\theta_1 s\theta_{234}$ –

We can solve Eqs. (2.159) a

$s\theta_6 = s\theta_5(u_x c\theta_1 c\theta_{234} + u_y s$

Equations (2.161) and (2.16

We conclude that there are a

2.7.4 Position Analysis

Figure 2.17 shows a 6-dof
(Scheinman, 1969). In this r
sliding pair) while all the o
up vertically in the positive
the first perpendicularly at p
third joint axis intersects th

where $k_1 = p_x c\theta_1 + p_y s\theta_1 - a_4 c\theta_{234}$ and $k_2 = p_z - a_4 s\theta_{234}$. Summing the squares of Eqs. (2.154) and (2.155), we obtain

$$a_2^2 + a_3^2 + 2a_2 a_3 c\theta_3 = k_1^2 + k_2^2. \tag{2.156}$$

Hence

$$\theta_3 = \cos^{-1} \frac{k_1^2 + k_2^2 - a_2^2 - a_3^2}{2a_2 a_3}. \tag{2.157}$$

Therefore, corresponding to each solution set of θ_1, θ_5, and θ_{234}, there are at most two real solutions of θ_3. Namely, if θ_3^* is a solution, $\theta_3 = -\theta_3^*$ is also a solution. Once θ_3 is known, θ_2 can be obtained by solving Eqs. (2.154) and (2.155) simultaneously for $s\theta_2$ and $c\theta_2$. This produces one solution of θ_2. Finally, the solutions of θ_4 are obtained from the relation $\theta_4 = \theta_{234} - \theta_2 - \theta_3$.

To solve for θ_6, we apply the transformation to the unit vector $\mathbf{u}$:

$$(R_1 R_2 R_3 R_4)^{\mathrm{T}} \mathbf{u} = R_5 R_6 \mathbf{u}_0. \tag{2.158}$$

Expanding Eq. (2.158), we obtain

$$u_x c\theta_1 c\theta_{234} + u_y s\theta_1 c\theta_{234} + u_z s\theta_{234} = s\theta_5 s\theta_6, \tag{2.159}$$

$$-u_x s\theta_1 + u_y c\theta_1 = -c\theta_5 s\theta_6, \tag{2.160}$$

$$-u_x c\theta_1 s\theta_{234} - u_y s\theta_1 s\theta_{234} + u_z c\theta_{234} = c\theta_6. \tag{2.161}$$

We can solve Eqs. (2.159) and (2.160) for $s\theta_6$:

$$s\theta_6 = s\theta_5 (u_x c\theta_1 c\theta_{234} + u_y s\theta_1 c\theta_{234} + u_z s\theta_{234}) - c\theta_5 (-u_x s\theta_1 + u_y c\theta_1). \tag{2.162}$$

Equations (2.161) and (2.162) together determine a unique solution for θ_6:

$$\theta_6 = \mathrm{Atan2}(s\theta_6, c\theta_6). \tag{2.163}$$

We conclude that there are at most eight real inverse kinematic solutions.

2.7.4 Position Analysis of the Stanford Arm

Figure 2.17 shows a 6-dof manipulator developed at Stanford University (Scheinman, 1969). In this manipulator, the third joint is a prismatic joint (or sliding pair) while all the others are revolute. The first joint axis, $\$_1$, points up vertically in the positive z-direction. The second joint axis, $\$_2$, intersects the first perpendicularly at point A and points in the positive x-direction. The third joint axis intersects the second perpendicularly at point B and points