

METR4202 -- Robotics

Tutorial 4 – Week 5: Robot Dynamics and Control

Solutions

Reading

Please read/review chapter 8 & 9 of Robotics, Vision and Control.

Review: Forward Kinematics of a two-link planar manipulator

$$\begin{aligned} x &= a_1 \cos \theta_1 + a_2 \cos(\theta_1 + \theta_2) \\ y &= a_1 \sin \theta_1 + a_2 \sin(\theta_1 + \theta_2) \end{aligned}$$

Questions

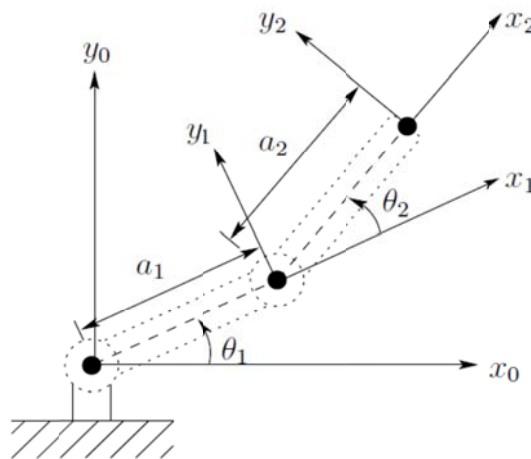


Figure 1: Two-link planar manipulator

1.

- a.) Using the two-link planar manipulator from tutorial 3, calculate the jacobian needed to relate the joint velocities to the tool-point velocities.

$$J = \begin{bmatrix} -l_1 \sin(\theta_1) - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

- b.) Similarly, calculate the inverse jacobian needed to relate the tool-point velocities to the joint velocities.

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \frac{1}{L_1 L_2 s_2} \begin{bmatrix} L_2 c_{12} & L_2 s_{12} \\ -L_1 c_1 - L_2 c_{12} & -L_1 s_1 - L_2 s_{12} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

2.

- a.) Using the Jacobian found in Q1a, calculate the tool point linear velocity if joint 1 is rotating at 1 rad/s and joint 2 is rotating at 3 rad/s ($a_1 = 2$, $a_2 = 3$, $\theta_1 = 167.028^\circ$, $\theta_2 = -156.44^\circ$).

$$\begin{aligned}\vec{v} &= J\vec{\dot{q}} \\ \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} &= \begin{bmatrix} -l_1 \sin(\theta_1) - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \\ \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} &= \begin{bmatrix} -1.0000 & -0.5512 \\ 1.0000 & 2.9489 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} &= \begin{bmatrix} -2.654 \\ 9.847 \end{bmatrix} m/s\end{aligned}$$

- b.) Calculate the resulting joint torques τ , given a force $F = (30, -20)$ is applied to the end effector tool point.

$$\begin{aligned}\tau &= J^T F \\ \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} &= \begin{bmatrix} -l_1 \sin(\theta_1) - l_2 \sin(\theta_1 + \theta_2) & l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) \\ -l_2 \sin(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \\ \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} &= \begin{bmatrix} -1.0000 & 1.0000 \\ -0.5512 & 2.9489 \end{bmatrix} \begin{bmatrix} 30 \\ -20 \end{bmatrix} \\ \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} &= \begin{bmatrix} -50.0 \\ -75.1 \end{bmatrix} Nm\end{aligned}$$

3. (See also p. 209 of Spong, *Robot Modeling and Control* [p. 17 of attached PDF] or Ex 13.13 (p.637) of LaValle, *Planning Algorithms* [p.772 of Ch. 13 of the [online PDF](#)], or p. 110 of Asada and Slotine, *Robot Analysis and Control*)

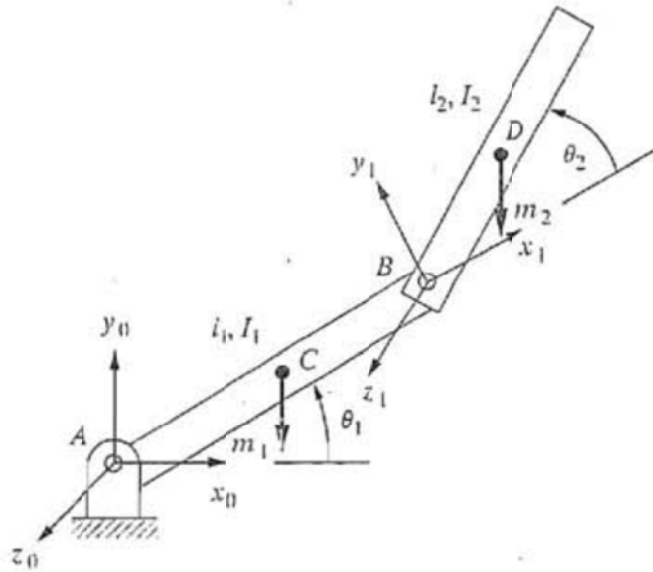


Figure 2: Two-link revolute joint arm.

a.) With respect to figure 2 above, derive the equations of motion for the two-degree-of-freedom robot arm using the Lagrangian method.

$$\tau_i = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_i} \right) - \frac{\partial L}{\partial \theta_i}$$

Step 1: Calculate the velocities at the center of mass of link 2.

$$x_D = l_1 C_1 + 0.5 l_2 C_{12} \Rightarrow \dot{x}_D = -l_1 S_1 \dot{\theta}_1 - 0.5 l_2 S_{12} (\dot{\theta}_1 + \dot{\theta}_2)$$

$$y_D = l_1 S_1 + 0.5 l_2 S_{12} \Rightarrow \dot{y}_D = l_1 C_1 \dot{\theta}_1 + 0.5 l_2 C_{12} (\dot{\theta}_1 + \dot{\theta}_2)$$

Total Velocity :

$$\begin{aligned} v_D^2 &= \dot{x}_D^2 + \dot{y}_D^2 \\ &= \dot{\theta}_1^2 (l_1^2 + 0.25 l_2^2 + l_1 l_2 C_2) + \dot{\theta}_2^2 (0.25 l_2^2) + \dot{\theta}_1 \dot{\theta}_2 (0.5 l_2^2 + l_1 l_2 C_2) \end{aligned}$$

Step 2: Calculate total Kinetic energy.

$$\begin{aligned} K &= K_1 + K_2 \\ &= \left[\frac{1}{2} I_A \dot{\theta}_1^2 \right] + \left[\frac{1}{2} I_D (\dot{\theta}_1 + \dot{\theta}_2)^2 + \frac{1}{2} m_2 v_D^2 \right] \end{aligned}$$

Substitute the total velocity into the Kinetic energy.

$$\begin{aligned} K &= \dot{\theta}_1^2 \left(\frac{1}{6} m_1 l_1^2 + \frac{1}{6} m_2 l_2^2 + \frac{1}{2} m_2 l_1^2 + \frac{1}{2} m_2 l_1 l_2 C_2 \right) + \dot{\theta}_2^2 \left(\frac{1}{6} m_2 l_2^2 \right) \\ &\quad + \dot{\theta}_1 \dot{\theta}_2 \left(\frac{1}{3} m_2 l_2^2 + \frac{1}{2} m_2 l_1 l_2 C_2 \right) \end{aligned}$$

Step 3: Calculate the total Potential energy of the system.

$$P = \frac{m_1 g l_1}{2} S_1 + m_2 g \left(l_1 S_1 + \frac{l_2}{2} S_{12} \right)$$

The Lagrangian for the two-link robot arm is:

$$\begin{aligned}
L &= K - P \\
&= \dot{\theta}^2 \left(\frac{1}{6} m_1 l_1^2 + \frac{1}{6} m_2 l_2^2 + \frac{1}{2} m_2 l_1^2 + \frac{1}{2} m_2 l_1 l_2 C_2 \right) + \dot{\theta}_2^2 \left(\frac{1}{6} m_2 l_2^2 \right) \\
&\quad + \dot{\theta}_1 \dot{\theta}_2 \left(\frac{1}{3} m_2 l_2^2 + \frac{1}{2} m_2 l_1 l_2 C_2 \right) - \frac{m_1 g l_1}{2} S_1 - m_2 g \left(l_1 S_1 + \frac{l_2}{2} S_{12} \right)
\end{aligned}$$

Step 4: Calculate the derivatives of the Lagrangian to determine the torque equations for the two-link robot arm: **Recall Chain rule expansion:**

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

$$\begin{aligned}
\frac{d}{dt} &= \frac{\partial}{\partial \theta_2} \frac{d\theta_2}{dt} + \frac{\partial}{\partial \dot{\theta}_1} \frac{d\dot{\theta}_1}{dt} \\
\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_1} \right) &= \left(\frac{1}{3} m_1 l_1^2 + m_2 l_1^2 + \frac{1}{3} m_2 l_2^2 + m_2 l_1 l_2 C_2 \right) \ddot{\theta}_1 \\
&\quad + \left(\frac{1}{3} m_2 l_2^2 + \frac{1}{2} m_2 l_1 l_2 C_2 \right) \ddot{\theta}_2 - (m_2 l_1 l_2 S_2) \dot{\theta}_1 \dot{\theta}_2 - \left(\frac{1}{2} m_2 l_1 l_2 S_2 \right) \dot{\theta}_2^2 \\
\frac{\partial L}{\partial \theta_1} &= \left(\frac{1}{2} m_1 + m_2 \right) g l_1 C_1 + \frac{1}{2} m_2 g l_2 C_{12}
\end{aligned}$$

$$\begin{aligned}
\tau_1 &= \left(\frac{1}{3} m_1 l_1^2 + m_2 l_1^2 + \frac{1}{3} m_2 l_2^2 + m_2 l_1 l_2 C_2 \right) \ddot{\theta}_1 + \left(\frac{1}{3} m_2 l_2^2 + \frac{1}{2} m_2 l_1 l_2 C_2 \right) \ddot{\theta}_2 - \\
&\quad (m_2 l_1 l_2 S_2) \dot{\theta}_1 \dot{\theta}_2 - \left(\frac{1}{2} m_2 l_1 l_2 S_2 \right) \dot{\theta}_2^2 + \left(\frac{1}{2} m_1 + m_2 \right) g l_1 C_1 + \frac{1}{2} m_2 g l_2 C_{12}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} &= \frac{\partial}{\partial \theta_2} \frac{d\theta_2}{dt} + \frac{\partial}{\partial \dot{\theta}_2} \frac{d\dot{\theta}_2}{dt} \\
\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_2} \right) &= \left(\frac{1}{3} m_2 l_2^2 + \frac{1}{2} m_2 l_1 l_2 C_2 \right) \ddot{\theta}_1 + \left(\frac{1}{3} m_2 l_2^2 \right) \ddot{\theta}_2 - \left(\frac{1}{2} m_2 l_1 l_2 S_2 \right) \dot{\theta}_1^2 \\
&\quad - (m_2 l_1 l_2 S_2) \dot{\theta}_1 \dot{\theta}_2 \\
\frac{\partial L}{\partial \theta_2} &= (m_2 l_1 l_2 S_2) \dot{\theta}_1 \dot{\theta}_2 + \frac{1}{2} m_2 g l_2 C_{12}
\end{aligned}$$

$$\tau_2 = \left(\frac{1}{3} m_2 l_2^2 + \frac{1}{2} m_2 l_1 l_2 C_2 \right) \ddot{\theta}_1 + \left(\frac{1}{3} m_2 l_2^2 \right) \ddot{\theta}_2 - \left(\frac{1}{2} m_2 l_1 l_2 S_2 \right) \dot{\theta}_1^2 + \frac{1}{2} m_2 g l_2 C_{12}$$

Challenge Question:

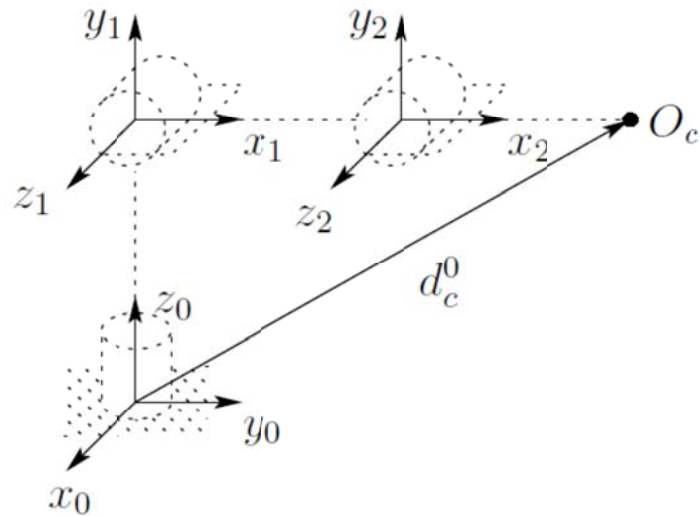


Figure 3: Elbow Manipulator

- List the DH parameters for this arm, clearly indicating which parameters are the joint variables ($L_1 = 3\text{m}$, $L_2 = 2\text{m}$, $L_3 = 1\text{m}$).
- Find the inverse Kinematic equations for the arm to derive the joint values from tool point position.
- Given that the tool point is at $(1.0\text{m}, 0.2\text{m}, 0.5\text{m})^T$, use the inverse kinematic equations to find the joint values.
- Find the manipulator Jacobian, J , that relates the joint velocities to the tool point velocity.