Chapter 4

The Jacobian

4.1 Introduction

We have thus far established the mathematical models for the forward kinematics and inverse kinematics of a manipulator. These models describe the relationships between the static configurations of a mechanism and its end-effector. The focus in this chapter is on the models associated with the velocities and static forces of articulated mechanisms and the Jacobian matrix which is central to these models.

Assuming the manipulator is at a given configuration, $\mathbf{q}$, let us imagine that all its joints undertook a set of infinitesimally small displacements, represented by the vector $\delta\mathbf{q}$. At the end-effector, there will be a corresponding set of displacements of the position and orientation $\mathbf{x}$, represented by the vector $\delta\mathbf{x}$. The goal in this chapter is to establish the relationship between $\delta\mathbf{x}$ and $\delta\mathbf{q}$. By considering the time derivatives of $\mathbf{x}$ and $\mathbf{q}$, this same relationship can be viewed as a relationship between the velocities $\dot{\mathbf{x}}$ and $\dot{\mathbf{q}}$. The relationship between $\dot{\mathbf{x}}$ and $\dot{\mathbf{q}}$ is described by the Jacobian matrix. Because of the duality between forces and velocities, this matrix as we will see later in this chapter is key to the relationship between joint torques and end-effector forces.

4.2 Differential Motion

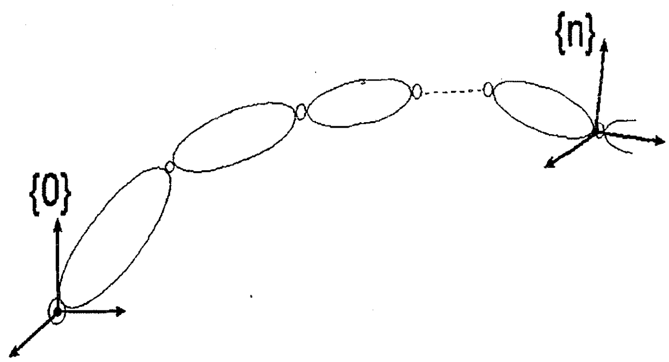

Figure 4.1: A Manipulator

Let us consider the function **f** that maps the space defined by variable **q** to the space defined by the variable **x**. Both **q** and **x** are vector variables (n and m- dimensional resp.), related by

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} f_1(q) \\ f_2(q) \\ \vdots \\ f_m(q) \end{pmatrix} \tag{4.1}$$

As described above we can consider the infinitesimal motion of the relationship $\mathbf{x} = \mathbf{f}(\mathbf{q})$. If we write it for each component of **x** and **q** we can derive the following set of equations for $\delta x_1, \delta x_2, \ldots, \delta x_m$ as functions of $\delta q_1, \delta q_2, \ldots, \delta q_n$

$$\delta x_1 = \frac{\partial f_1}{\partial q_1}\delta q_1 + \cdots + \frac{\partial f_1}{\partial q_n}\delta q_n \tag{4.2}$$

$$\begin{aligned} \vdots &= \vdots \\ \delta x_m &= \frac{\partial f_m}{\partial q_1}\delta q_1 + \cdots + \frac{\partial f_m}{\partial q_n}\delta q_n \end{aligned} \tag{4.3}$$

The above equations can be written in vector form as follows

$$\delta \mathbf{x} = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \cdots & \frac{\partial f_1}{\partial q_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial q_1} & \cdots & \frac{\partial f_m}{\partial q_n} \end{bmatrix} \delta \mathbf{q} \tag{4.4}$$

The matrix in the above relationship is called the Jacobian matrix and is function of $\mathbf{q}$.

$$J(\mathbf{q}) \equiv \frac{\partial \mathbf{f}}{\partial \mathbf{q}} \tag{4.5}$$

In general, the Jacobian allows us to relate corresponding small displacements in different spaces. If we divide both sides of the relationship by small time interval (i.e. differentiate with respect to time) we obtain a relationship between the velocities of the mechanism in joint and Cartesian space.

$$\dot{\mathbf{x}}_{(m\times 1)} = J(\mathbf{q})_{m\times n}\dot{\mathbf{q}}_{(n\times 1)} \tag{4.6}$$

4.2.1 Example: RR Manipulator

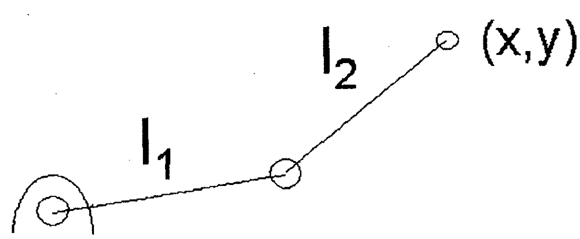

Figure 4.2: A 2 link example

The Jacobian is a mxn matrix from its definition. To illustrate the Jacobian, let us consider the following example. Take a two link manipulator in the plane with revolute joints and axis of rotation perpendicular to the plane of the paper. Let us first derive the positional part of a Jacobian. First from the forward kinematics we derive the description of the position and orientation of the end-effector in Cartesian space with respect to the joint coordinates θ_1 and θ_2.

$$x = l_1 c_1 + l_2 c_{12} \tag{4.7}$$

$$y = l_1 s_1 + l_2 s_{12} \tag{4.8}$$

The instantaneous motion of the position vector (x, y) is

$$\delta x = -(l_1 s_1 + l_2 s_{12})\delta\theta_1 - l_2 s_{12}\delta\theta_2 \tag{4.9}$$

$$\delta y = (l_1 c_1 + l_2 c_{12})\delta\theta_1 + l_2 c_{12}\delta\theta_2 \tag{4.10}$$

If we group the coefficients in front of $\delta\theta_1$ and $\delta\theta_2$ we obtain a matrix equation which can be written as

$$\delta x = \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = \begin{bmatrix} -y & -l_2 s_{12} \\ x & l_2 c_{12} \end{bmatrix} \begin{pmatrix} \delta\theta_1 \\ \delta\theta_2 \end{pmatrix} \tag{4.11}$$

The $2x2$ matrix in the above equation is the Jacobian, $J(\mathbf{q})$.

$$\delta\mathbf{x} = J(\mathbf{q})\delta\mathbf{q} \tag{4.12}$$

As we can see this matrix is a function of the vector $\mathbf{q} = (\theta_1, \theta_2)$.

$$J \equiv \begin{pmatrix} \frac{\partial x}{\partial\theta_1} & \frac{\partial x}{\partial\theta_2} \\ \frac{\partial y}{\partial\theta_1} & \frac{\partial y}{\partial\theta_2} \end{pmatrix} \tag{4.13}$$

Now if we consider the differentiation *w.r.t.* time, we can write the relationship between $\dot{\mathbf{x}}$ and $\dot{\mathbf{q}}$.

$$\dot{\mathbf{x}} = J(\mathbf{q})\dot{\mathbf{q}} \tag{4.14}$$

4.2.2 Example: Stanford Scheinman Arm

As another example, we describe below the Jacobian associated with the end-effector position of the Stanford Scheinman arm . The first three joint variables here are θ_1, θ_2 and d_3. From the forward kinematics we can observe that the position of the end-effector as a function of θ_1, θ_2 and d_3 is:

$$\mathbf{x}_p = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 \\ s_1 s_2 d_3 + c_1 d_2 \\ c_2 d_3 \end{bmatrix} \tag{4.15}$$

If we differentiate with respect to the joint vector $(\theta_1, \theta_2, d_3, \theta_4, \theta_5, \theta_6)$ we obtain the following Jacobian for the position of the end-effector.

$$\dot{\mathbf{x}}_p = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{bmatrix} -y & c_1 c_2 d_3 & c_1 s_2 & 0 & 0 & 0 \\ x & s_1 c_2 d_3 & s_1 s_2 & 0 & 0 & 0 \\ 0 & -s_2 d_3 & c_2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix} \tag{4.16}$$

We defined the position part as $\mathbf{x}_p$ and the corresponding part of the Jacobian will be denoted as J_p.

$$\dot{\mathbf{x}}_{p(3\times1)} = J_{p(3\times6)}(\mathbf{q})\dot{\mathbf{q}}_{(6\times1)} \tag{4.17}$$

For the orientation we will derive a Jacobian associated with the end-effector orientation representation, $\mathbf{x}_r$.

$$\dot{\mathbf{x}}_r = J_r(\mathbf{q})\dot{\mathbf{q}} \tag{4.18}$$

In our example the orientation part is given in terms of direction cosines $(r_{11}, r_{12}, \ldots, r_{33})$. When we differentiate those *w.r.t.* the joint variables, we will obtain the Jacobian for this orientation representation.

$$\mathbf{x}_r = \begin{bmatrix} \mathbf{r}_1(q) \\ \mathbf{r}_2(q) \\ \mathbf{r}_3(q) \end{bmatrix} \tag{4.19}$$

$$\dot{\mathbf{x}}_r = \begin{pmatrix} \dot{\mathbf{r}}_1 \\ \dot{\mathbf{r}}_2 \\ \dot{\mathbf{r}}_3 \end{pmatrix}_{(9\times1)} = \begin{pmatrix} \frac{\partial \mathbf{r}_1}{\partial q_1} & \cdots & \frac{\partial \mathbf{r}_1}{\partial q_6} \\ \frac{\partial \mathbf{r}_2}{\partial q_1} & \cdots & \frac{\partial \mathbf{r}_2}{\partial q_6} \\ \frac{\partial \mathbf{r}_3}{\partial q_1} & \cdots & \frac{\partial \mathbf{r}_3}{\partial q_6} \end{pmatrix}_{(9\times6)} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_6 \end{pmatrix}_{(6\times1)} \tag{4.20}$$

This is a 9×6 matrix because we are using the redundant direction cosines representation for the orientation. As time derivatives the relationship between $\dot{\mathbf{q}}$ and $\dot{\mathbf{x}}_r$ (the derivative of the orientation) is described by J_r (Jacobian of the orientation). Finally we can put the position and the orientation part together below.

$$\dot{\mathbf{x}}_p = J_p(\mathbf{q})\dot{\mathbf{q}} \tag{4.21}$$

$$\dot{\mathbf{x}}_r = J_r(\mathbf{q})\dot{\mathbf{q}} \tag{4.22}$$

The above equations can be combined as

$$\begin{pmatrix} \dot{\mathbf{x}}_p \\ \dot{\mathbf{x}}_r \end{pmatrix} = \begin{pmatrix} J_p(\mathbf{q}) \\ J_r(\mathbf{q}) \end{pmatrix} \dot{\mathbf{q}} \tag{4.23}$$

We can see that this Jacobian is a 12×6 matrix.

$$\dot{\mathbf{x}}_{(12\times1)} = J_x(\mathbf{q})_{(12\times6)}\dot{\mathbf{q}}_{(6\times1)} \tag{4.24}$$

We should also note that so far we have not used any explicit frame in which we are describing those quantities, i.e. these equations are valid for any common frame that the variables are described in.

The above matrix is clearly dependent on the end effector representation. If we have selected a different representation for the orientation or the position of the end-effector we will obtain a different Jacobian matrix.

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_p \\ \mathbf{x}_r \end{bmatrix}$$

$$\mathbf{x} = \mathbf{f}(\mathbf{q}) \longrightarrow J_x = \frac{\partial \mathbf{f}(\mathbf{q})}{\partial \mathbf{q}}$$

Typically the position, $\mathbf{x}_p$ is represented by the three Cartesian coordinates of a point on the end- effector (x, y, z). However we can also use spherical or cylindrical coordinates for that end-effector point and this will lead to a different Jacobian J_p. The orientation can be also described by different sets of parameters - Euler angles, direction cosines, Euler parameter, equivalent axis parameters, etc.

Depending on the representation used we will have different dimension of the orientation component of the Jacobian - $3 \times n$ for Euler angles, $9 \times n$ for direction cosines, $4 \times n$ for Euler parameters or equivalent axis parameters, where n is the number of degrees of freedom of the mechanism.

4.3 Basic Jacobian

We will introduce a unique Jacobian that is associated with the motion of the mechanism.

As we mentioned earlier, the Jacobian we have talked so far about depends on the representation used for the position and orientation of the end-effector.

If we use spherical coordinates for the position and direction cosines for the orientation we will obtain one Jacobian (12 for 6 DOF robot) very different from the one that results from Cartesian coordinates for the position and Euler parameters for the orientation (7×6 matrix for a 6 DOF robot).

Defined from the differentiation of $\mathbf{x} = \mathbf{f}(\mathbf{q})$ with respect to $\mathbf{q}$, the Jacobian is dependent on the representation $\mathbf{x}$ of the end-effector position and orientation. Since the kinematic properties of a mechanism are independent of the selected representation, it is important for the kinematic model to also be representation-independent. The Jacobian associated with such a model is unique. This Jacobian will be called the basic Jacobian.

The basic Jacobian matrix establishes the relationships between joint velocities and the corresponding (uniquely-defined) linear and angular velocities at a given point on the end-effector.

$$\begin{pmatrix} \mathbf{v} \\ \omega \end{pmatrix}_{(6\times 1)} = J_0(q)_{(6\times n)}\dot{\mathbf{q}}_{(n\times 1)} \tag{4.25}$$

Linear velocities are the time derivatives of the Cartesian coordinates of the end-effector position vector. However this is not the case for any orientation representation. For example if we take (α, β, γ) Euler angles, their derivatives are not the angular velocities. In fact angular velocities do not have a primitive function, no representation of the orientation has derivatives equal to the angular velocities. The angular velocity is defined as an instantaneous quantity. However, the time derivative of any representation of the orientation is related to the angular velocity. This is also the case for general position representation. These relationships are of the form

$$\dot{\mathbf{x}}_p = E_p(\mathbf{x}_p)\mathbf{v} \tag{4.26}$$

$$\dot{\mathbf{x}}_r = E_r(\mathbf{x}_r)\omega \tag{4.27}$$

Here $\dot{x}_p$ is the time derivative of the position part of the end-effector representation and $\dot{x}_r$ is the time derivative of the orientation part. The matrices E_p and E_r are only dependent on the particular position or orientation representation of the end-effector. Using E_p and E_r we will be able to obtain the Jacobian for the particular representation as a function of the basic Jacobian.

4.3.1 Example: E_p, E_r

As an illustration, if for example we use Cartesian coordinates for the end-effector position and $\alpha - \beta - \gamma$ Euler angles for the end-effector orientation

$$\mathbf{x}_p = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{4.28}$$

$$\mathbf{x}_r = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \tag{4.29}$$

the corresponding matrices for E_p and E_r are:

$$E_p(\mathbf{x}_p) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.30}$$

$$E_r(\mathbf{x}_r) = \begin{pmatrix} -\frac{s\alpha.c\beta}{s\beta} & \frac{c\alpha.c\beta}{s\beta} & 1 \\ c\alpha & s\alpha & 0 \\ \frac{s\alpha}{s\beta} & -\frac{c\alpha}{s\beta} & 0 \end{pmatrix} \tag{4.31}$$

As mentioned earlier E_p is the unit 3×3 matrix for that example.

4.3.2 Relationship: J_x and J_O

The basic Jacobian, J_0, is defined as

$$\begin{pmatrix} \mathbf{v} \\ \omega \end{pmatrix} = J_0(\mathbf{q})\dot{\mathbf{q}} \tag{4.32}$$

We will denote J_v and J_ω as the linear and angular velocity parts of this matrix.

$$\begin{cases} \mathbf{v} = J_v\dot{\mathbf{q}} \\ \omega = J_\omega\dot{\mathbf{q}} \end{cases} \tag{4.33}$$

Using the definitions of E_p and E_r above

$$\dot{\mathbf{x}}_p = E_p \mathbf{v} \Rightarrow \dot{\mathbf{x}}_p = (E_p J_v)\dot{\mathbf{q}} \tag{4.34}$$

and

$$\dot{\mathbf{x}}_r = E_r \omega \Rightarrow \dot{\mathbf{x}}_r = (E_r J_\omega)\dot{\mathbf{q}} \tag{4.35}$$

we can derive the following relationships between J_p and J_r and the basic Jacobian's components J_v and J_w.

$$\begin{cases} J_{X_P} = E_P J_v \\ J_{X_R} = E_R J_\omega \end{cases} \tag{4.36}$$

The above relationships can also be arranged in a matrix form by introducing the matrix $E_{(6\times 6)}$

$$J_x = \begin{pmatrix} J_p \\ J_r \end{pmatrix} = \begin{pmatrix} E_p & 0 \\ 0 & E_r \end{pmatrix} \begin{pmatrix} J_v \\ J_\omega \end{pmatrix} \tag{4.37}$$

Using E, the relationship between J_x and the basic Jacobian J_0 becomes

$$J_x(\mathbf{q}) = E(\mathbf{x}) J_0(\mathbf{q}) \tag{4.38}$$

with

$$\begin{pmatrix} \mathbf{v} \\ \omega \end{pmatrix} = J_0(q)\dot{\mathbf{q}} \tag{4.39}$$

For the example above

$$E_p = I_3; J_p = J_v \tag{4.40}$$

and

$$E = \begin{pmatrix} I & 0 \\ 0 & E_r \end{pmatrix} \tag{4.41}$$

4.4 Linear/Angular Motion

In this section we further analyze the linear and angular velocities associated with multi-body systems. Let us consider a point P described by a position vector $\mathbf{p}$ with respect to the origin of a fixed frame $\{A\}$. If the point P is moving with respect to frame $\{A\}$, the linear velocity of the point P with respect to frame $\{A\}$ is the vector $\mathbf{v}_{P/A}$. As a vector, the linear velocity can be expressed in any frame - $\{A\}, \{B\}, \{C\}$ with the coordinates ${}^A\mathbf{v}_{P/A}$, ${}^B\mathbf{v}_{P/A}$, ${}^C\mathbf{v}_{P/A}$. The relationships between these coordinates, involve the rotation transformation matrices introduced earlier. Naturally if the point P is fixed in frame $\{A\}$, the linear velocity vector of P with respect to $\{A\}$ will be zero.

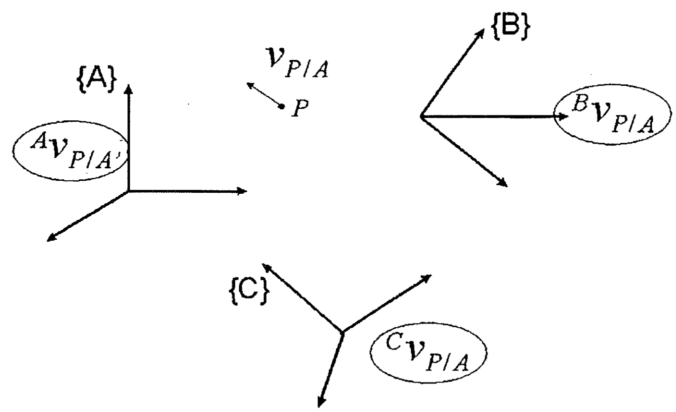

Figure 4.3: Linear Velocity

4.4.1 Pure Translation

Let us now consider a pure translation of frame $\{A\}$ with respect to another frame $\{B\}$. The linear velocity of point P with respect to {B} is $\mathbf{v}_{P/B}$. If $\mathbf{v}_{A/B}$ represents the velocity of the origin of frame $\{A\}$ with respect to frame $\{B\}$, the two vectors of linear velocities of point P with respect to {A} and {B} are related by

$$v_{P/B} = v_{A/B} + v_{P/A}$$

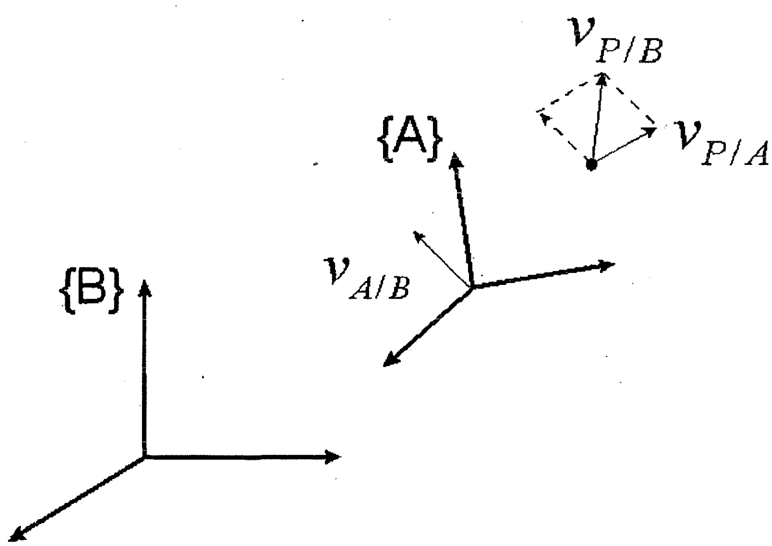

Figure 4.4: Pure Translation

4.4.2 Pure Rotation

To analyze the rotation of a rigid body, we need to define a point fixed in the body and an axis of rotation passing through this point. The body rotates about this axis and all the points along this axis are fixed *w.r.t.* this rotation. This rotation is described by a quantity called angular velocity, represented by the vector Ω.

A point P on the rotating rigid body is moving with a linear velocity $\mathbf{v}_P$, which is dependent on the magnitude of Ω and on the location of P with respect to the axis of rotation.

Different points on the rigid body will have different linear velocities. If we select a point O in the body along the axis of rotation the position vector $\mathbf{p}$ measured from O to P will be perpendicular to the linear

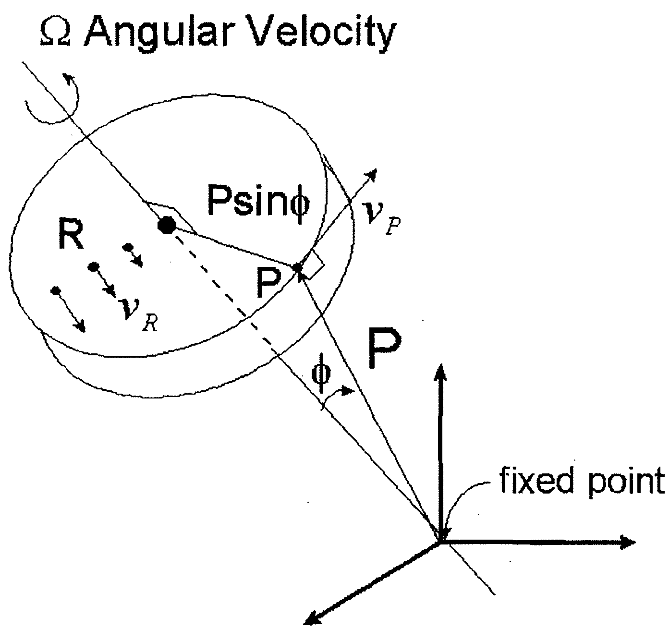

Figure 4.5: Rotational Motion

velocity vector $\mathbf{v}_p$. In addition from mechanics we know that the vector $\mathbf{v}_p$ is also perpendicular to the axis of rotation and in particular to Ω (the angular velocity vector). The magnitude of $\mathbf{v}_p$ is proportional to the magnitude of Ω (the rate of rotation) and to the distance to the axis of rotation, in other words to the magnitude of $\mathbf{p}\sin(\phi)$, as illustrated in Figure 4.5. Here ϕ is the angle between the axis of rotation and the position vector $\mathbf{p}$. Thus we can derive the following relationship

$$\mathbf{v}_P = \Omega \times \mathbf{p} \tag{4.42}$$

Using the definition of cross product operator, the above vector relationship can be described in the matrix form as

$$\mathbf{v}_P = \Omega \times \mathbf{p} \Rightarrow \mathbf{v}_P = \widehat{\Omega}\mathbf{p} \tag{4.43}$$

For instance. let us consider the components of vectors, Ω and $\mathbf{p}$.

$$\Omega = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} \quad \text{and} \quad \mathbf{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \tag{4.44}$$

With the cross product operator, the linear velocity of a point P is

$$\mathbf{v}_P = \widehat{\Omega}\mathbf{p} = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \tag{4.45}$$

4.4.3 Cross Product Operator and Rotation Matrix

Consider the rotation matrix between a frame fixed with respect to the rigid axis and frame moving with the rotated body. The cross product operator $\widehat{\Omega}$ can be expressed in terms of this rotation matrix.

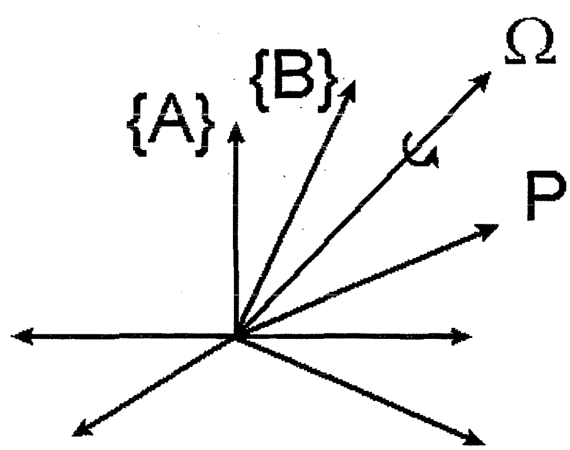

Figure 4.6: Rotation and Cross Product

Consider a pure rotation about an axis with an angular velocity Ω. Let P be a point fixed in body B. Then the velocity of P in B is zero, i.e.

$$\mathbf{v}_{P/B} = 0 \tag{4.46}$$

The representations ${}^B\mathbf{p}$ and ${}^A\mathbf{p}$ of the position vector $\mathbf{p}$ in frames $\{A\}$ and $\{B\}$ are related by the rotation matrix ${}^A_BR$

$${}^A\mathbf{p} = {}^A_BR\,{}^B\mathbf{p} \tag{4.47}$$

Let us differentiate *w.r.t.* time the above relationship

$${}^A\dot{\mathbf{p}} = {}^A_B\dot{R}\,{}^B\mathbf{p} + {}^A_BR\,{}^B\dot{\mathbf{p}}$$

Noting that the second term is equal to zero (since $\mathbf{v}_{P/B} = 0$), the relationship becomes

$${}^A\dot{\mathbf{p}} = {}^A_B\dot{R}\,{}^B\mathbf{p}$$

Transforming ${}^B\mathbf{p}$ to ${}^A\mathbf{p}$ by pre-multiplication of ${}^A_BR^T\,{}^A_BR = I$, yields

$$\begin{aligned} {}^A\dot{\mathbf{p}} &= {}^A_B\dot{R}\ (I)\ {}^B\mathbf{p} = {}^A_B\dot{R}\ ({}^A_BR^T\,{}^A_BR)\ {}^B\mathbf{p} & (4.48)\\ {}^A\dot{\mathbf{p}} &= {}^A_B\dot{R}\,{}^A_BR^T({}^A_BR\,{}^B\mathbf{p}) = ({}^A_B\dot{R}\,{}^A_BR^T)\,{}^A\mathbf{p} & (4.49) \end{aligned}$$

The above relationship can be written in vector form for any rotating frame

$$\dot{\mathbf{p}} = \dot{R}R^T\mathbf{p} \tag{4.50}$$

Observing that $\dot{\mathbf{p}}$ is linear velocity of $\mathbf{v}_P$, we obtain

$$\widehat{\Omega} = \dot{R}R^T \tag{4.51}$$

4.4.4 Example: Rotation About Axis Z

Consider the rotation of frame about the axis Z of a fixed frame. Measured by the angle θ, the corresponding rotation matrix is

$$R = \begin{pmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.52}$$

The derivative *w.r.t.* time is

$$\dot{R} = \begin{pmatrix} -s\theta\dot{\theta} & -c\theta\dot{\theta} & 0 \\ c\theta\dot{\theta} & -s\theta\dot{\theta} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.53}$$

or

$$\dot{R}.R^T = \begin{pmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.54}$$

Clearly vector ω here is just

$$\Omega = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta} \end{pmatrix} \tag{4.55}$$

and we can verify that

$$\widehat{\Omega} = \begin{pmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.56}$$

Thus the relationship above is verified.

$$\widehat{\Omega} = \dot{R}R^T \tag{4.57}$$

4.5 Combined Linear and Angular Motion

Now we consider motions involving both linear and angular velocities, as illustrated in Figure 4.7.

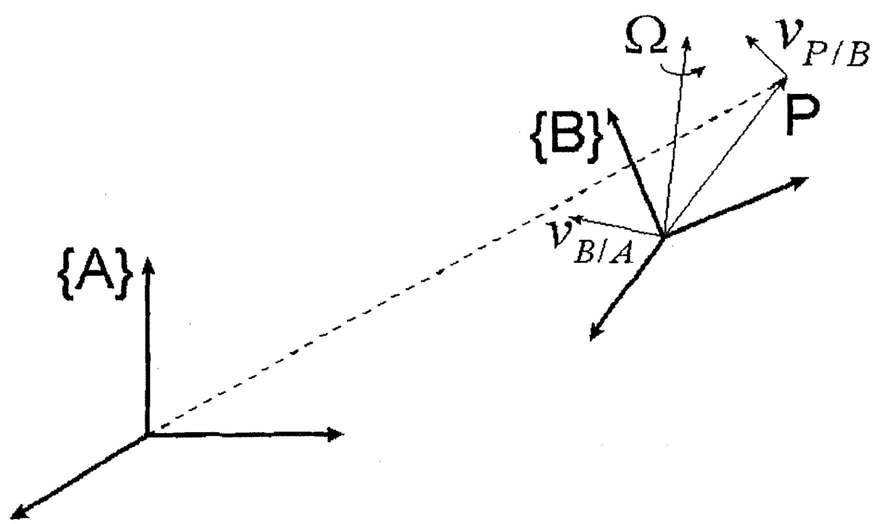

Figure 4.7: Linear and Angular Motion

The corresponding relationship is:

$$\mathbf{v}_{P/A} = \mathbf{v}_{B/A} + \mathbf{v}_{P/B} + \Omega \times \mathbf{p}_B \tag{4.58}$$

In order to perform this addition we need to have all quantities expressed in the same reference frame. In frame $\{A\}$ the equation is

$$^{A}\mathbf{v}_{P/A} = {}^{A}\mathbf{v}_{B/A} + {}^{A}_{B}R^{B}\mathbf{v}_{P/B} + {}^{A}\Omega_B \times {}^{A}_{B}R^{B}\mathbf{p}_B \tag{4.59}$$

4.6 Jacobian: Velocity Propagation

When we have several rigid bodies connected in a mechanism, we need to propagate the velocities from frame $\{0\}$ to frame $\{n\}$.

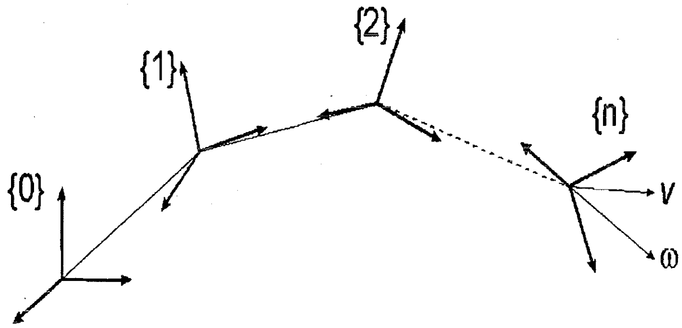

Figure 4.8: A Spatial Mechanism

The linear and angular velocity at the end-effector can be computed by propagation of velocities through the links of the manipulator. By computing and propagating linear and angular velocities from the fixed base to the end-effector, we establish the relationship between joint velocities and end- effector velocities. This provides an iterative method to compute the Basic Jacobian.

Consider two consecutive links i and $i+1$.

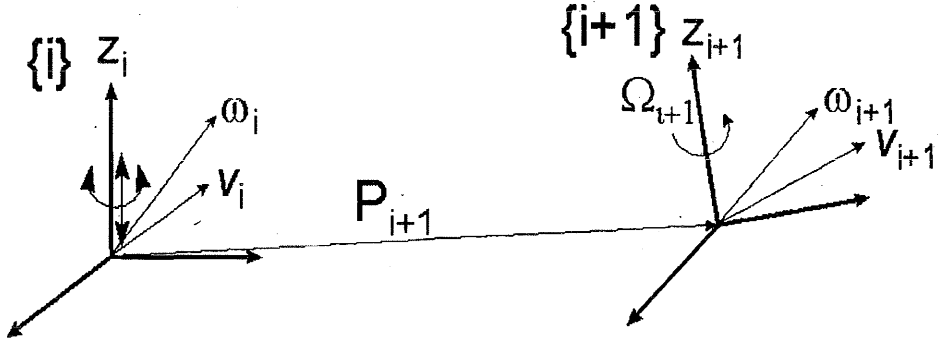

Figure 4.9: Velocity Propagation

The angular velocity of link $i+1$ is equal to the angular velocity of link i plus the local rotation of link $i+1$ represented by Ω_{i+1}.

$$\omega_{i+1} = \omega_i + \Omega_{i+1} \tag{4.60}$$

This local rotation is simply given by the derivative $\dot{\theta}_{i+1}$ of the angle of rotation of the link along the axis of rotation $\mathbf{z}_{i+1}$.

$$\Omega_{i+1} = \dot{\theta}_{i+1}\mathbf{z}_{i+1} \tag{4.61}$$

For the linear velocity the expression is slightly more complicated. The linear velocity at link $i+1$ is equal to the one at link i plus the contribution of the angular velocity of link i ($\omega_i \times \mathbf{p}_{i+1}$) plus the contribution of the local linear velocity associated with a prismatic joint (this is $\dot{d}_{i+l}\mathbf{z}_{i+1}$) if joint $i+1$ was prismatic.

$$\mathbf{v}_{i+1} = \mathbf{v}_i + \omega_i \times \mathbf{p}_{i+1} + \dot{d}_{i+1}\mathbf{z}_{i+1} \tag{4.62}$$

If we use these equations we can propagate them from the beginning to the end of the chain. If the computation of velocities is done in the local frame, the result will be obtained in frame $\{n\}$. The end-effector linear and angular velocities in the base frame are

$$\begin{pmatrix} {}^0v \\ {}^0\omega \end{pmatrix} = \begin{pmatrix} {}^0_nR & 0 \\ 0 & {}^0_nR \end{pmatrix} \begin{pmatrix} {}^nv \\ {}^n\omega \end{pmatrix} \tag{4.63}$$

The above expressions are linear functions of $\dot{\mathbf{q}}$, from which the basic Jacobian can be extracted. This iterative procedure is suitable for numerical computations of the Jacobian. The procedure, however, does not provide a description of the special structure of the Jacobian matrix. The next section addresses this aspect and presents a method for an explicit form of the Jacobian.

4.7 Jacobian: Explicit Form

Consider a general mechanism and let us examine how the velocities at the joints affect the linear and angular velocities at the end-effector.

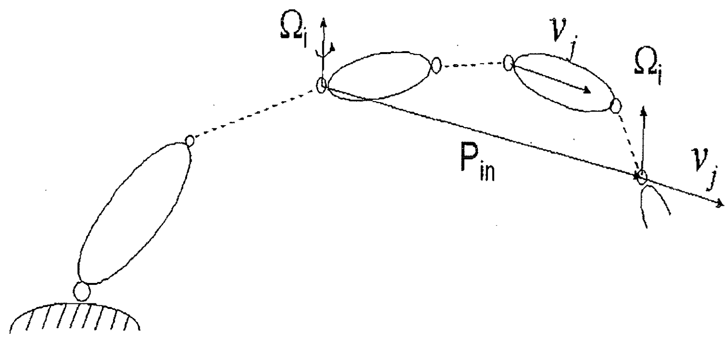

Figure 4.10: Explicit Form of the Jacobian

The velocity of a link with respect to the proceeding link is dependent on the type of joint that connects them. If the joint is a prismatic one, then the link linear velocity with respect to the previous link is along the prismatic joint axis, $\mathbf{z}_i$ with a magnitude of $\dot{q}_i$.

$$\mathbf{v}_i = \mathbf{z}_i \dot{q}_i \tag{4.64}$$

Similarly for a revolute joint the angular velocity is about the revolute joint axis with a magnitude of $\dot{q}_i$.

$$\Omega_i = \mathbf{z}_i \dot{q}_i \tag{4.65}$$

The local velocity at each joint contributes to the end effector velocities. A revolute joint creates both an angular rotation at the end-effector and a linear velocity. The linear velocity depends on the distance between the end-effector point and the joint axis. It involves the cross product of Ω_i with the vector locating this point. The angular velocity, Ω_i is transfered down the chain to the end-effector. A prismatic joint j creates only a linear velocity $\mathbf{v}_j$ that gets transfered down to the end-effector.

The total contribution of joint velocities of the mechanism to the end-effector linear velocity is therefore

$$\mathbf{v} = \sum_{i=1}^{n} [\epsilon_i v_i + \bar{\epsilon}_i(\Omega_i \times \mathbf{p}_{in})] \tag{4.66}$$

Similarly the end-effector angular velocity is the sum

$$\omega = \sum_{i=1}^{n} \bar{\epsilon}_i \Omega_i \tag{4.67}$$

Substituting the expressions of $\mathbf{v}_i$ and Ω_i from equations 4.64 and 4.65, we obtain

$$\mathbf{v} = \sum_{i=1}^{n} [\epsilon_i \mathbf{z}_i + \bar{\epsilon}_i(\mathbf{z}_i \times \mathbf{p}_{in})]\dot{q}_i \tag{4.68}$$

$$\omega = \sum_{i=1}^{n} \bar{\epsilon}_i \mathbf{z}_i \dot{q}_i \tag{4.69}$$

The end-effector velocity is:

$$\mathbf{v} = (\epsilon_1 \mathbf{z}_1 + \bar{\epsilon}_1(\mathbf{z}_1 \times \mathbf{p}_{1n}))\dot{q}_1 + (\epsilon_2 \mathbf{z}_2 + \bar{\epsilon}_2(\mathbf{z}_2 \times \mathbf{p}_{2n}))\dot{q}_2 + \cdots \tag{4.70}$$

or

$$\mathbf{v} = \begin{bmatrix} (\epsilon_1 \mathbf{z}_1 + \bar{\epsilon}_1(\mathbf{z}_1 \times \mathbf{p}_{1n})) & (\epsilon_2 \mathbf{z}_2 + \bar{\epsilon}_2(\mathbf{z}_2 \times \mathbf{p}_{2n})) & \cdots \end{bmatrix} . \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix} \tag{4.71}$$

and it can be written as:

$$\mathbf{v} = J_v \dot{\mathbf{q}} \tag{4.72}$$

where J_v is the linear motion Jacobian. Similarly the end-effector angular velocity is:

$$\omega = \bar{\epsilon}_1 \mathbf{z}_1 \dot{q}_1 + \bar{\epsilon}_2 \mathbf{z}_2 \dot{q}_2 + \cdots + \bar{\epsilon}_n \mathbf{z}_n \dot{q}_n \tag{4.73}$$

or

$$\omega = \begin{bmatrix} \bar{\epsilon}_1 \mathbf{z}_1 & \bar{\epsilon}_2 \mathbf{z}_2 & \cdots & \bar{\epsilon}_n \mathbf{z}_n \end{bmatrix} . \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix} \tag{4.74}$$

and it can be written as:

$$\omega = J_\omega \dot{\mathbf{q}} \tag{4.75}$$

where J_ω is the angular motion Jacobian. Combining the linear and angular motion parts leads to the basic Jacobian

$$\left\{ \begin{matrix} \mathbf{v} = J_v \dot{q} \\ \omega = J_\omega \dot{\mathbf{q}} \end{matrix} \right\} \rightarrow \begin{pmatrix} \mathbf{v} \\ \omega \end{pmatrix} = J \dot{\mathbf{q}} \tag{4.76}$$

or

$$J = \begin{pmatrix} J_v \\ J_\omega \end{pmatrix} \tag{4.77}$$

The equations provide the expressions for the matrices J_v and J_w. The derivation of the matrix J_v involves new quantities $\mathbf{p}_{1n}, \mathbf{p}_{2n}, \ldots, \mathbf{p}_{nn}$ that need to be computed.

A simple approach to compute J_v is to use the direct differentiation of the Cartesian coordinates of the point on the end-effector

$$\mathbf{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \dot{\mathbf{x}}_P = \frac{\partial x_P}{\partial q_1} \dot{q}_1 + \frac{\partial x_P}{\partial q_2} \dot{q}_2 + \cdots + \frac{\partial x_P}{\partial q_n} \dot{q}_n \tag{4.78}$$

$$\mathbf{v} = \begin{pmatrix} \frac{\partial x_P}{\partial q_1} & \frac{\partial x_P}{\partial q_2} & \cdots & \frac{\partial x_P}{\partial q_n} \end{pmatrix} \dot{\mathbf{q}} = J_v \dot{\mathbf{q}} \tag{4.79}$$

For J_w all that we need is to compute the $\mathbf{z}-$vectors associated with revolute joints. Overall the Jacobian takes the form

$$J = \begin{pmatrix} \frac{\partial x_P}{\partial q_1} & \frac{\partial x_P}{\partial q_2} & \cdots & \frac{\partial x_P}{\partial q_n} \\ \bar{\epsilon}_1 \mathbf{z}_1 & \bar{\epsilon}_2 \mathbf{z}_2 & \cdots & \bar{\epsilon}_n \mathbf{z}_n \end{pmatrix} \tag{4.80}$$

Note that ϵ is zero for a revolute joint and one for a prismatic one. To express the Jacobian in particular frame, all we need is to have all the quantities expressed in that frame.

$${}^0J = \begin{pmatrix} \frac{\partial {}^0x_P}{\partial q_1} & \frac{\partial {}^0x_P}{\partial q_2} & \cdots & \frac{\partial {}^0x_P}{\partial q_n} \\ \bar{\epsilon}_1 {}^0\mathbf{z}_1 & \bar{\epsilon}_2 {}^0\mathbf{z}_2 & \cdots & \bar{\epsilon}_n {}^0\mathbf{z}_n \end{pmatrix} \tag{4.81}$$

The components of ${}^0\mathbf{z}_i$ can be found as ${}^0\mathbf{z}_i = {}^0_iR^i\mathbf{z}_i$ (${}^i\mathbf{z}_i$ is of course $(0 \quad 0 \quad 1)$). Thus all we need for the angular motion Jacobian is the last column of the rotation matrix.

$$\begin{pmatrix} X \\ X \\ X \end{pmatrix} = \begin{pmatrix} & & X \\ & & X \\ & & X \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{4.82}$$

$${}^0\mathbf{z}_i = {}^0_iR\mathbf{z} \tag{4.83}$$

with

$$\mathbf{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{4.84}$$

The overall Jacobian is then found as:

$${}^0J = \begin{pmatrix} \frac{\partial}{\partial q_1}({}^0x_P) & \frac{\partial}{\partial q_2}({}^0x_P) & \cdots & \frac{\partial}{\partial q_n}({}^0x_P) \\ \bar{\epsilon}_1({}^0_1R\mathbf{z}) & \bar{\epsilon}_2({}^0_2R\mathbf{z}) & \cdots & \bar{\epsilon}_n({}^0_nR\mathbf{z}) \end{pmatrix} \tag{4.85}$$

4.7.1 Example: Stanford Scheinman Arm

As we have shown previously, we first introduce frames, define the D&H parameters and calculate the D&H table. Then we calculate the transformation matrices, namely:

$$ {}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.86} $$

$$ {}^1_2T = \begin{bmatrix} c_2 & -s_2 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ -s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.87} $$

$$ {}^2_3T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -d_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.88} $$

$$ {}^3_4T = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.89} $$

$$ {}^4_5T = \begin{bmatrix} c_5 & -s_5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_5 & -c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.90} $$

$$ {}^5_6T = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.91} $$

Next we express each of the frames *w.r.t.* the $\{0\}$ frame, i.e. we calculate the transformation matrices:

$$ {}^0_2T = \begin{bmatrix} c_1c_2 & -c_1s_2 & -s_1 & -s_1d_2 \\ s_1c_2 & -s_1s_2 & c_1 & c_1d_2 \\ -s_2 & -c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.92} $$

$$ {}^0_3T = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & c_1d_3s_2 - s_1d_2 \\ s_1c_2 & c_1 & s_1s_2 & s_1d_3s_2 + c_1d_2 \\ -s_2 & 0 & c_2 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.93} $$

$$ {}^0_4T = \begin{bmatrix} c_1c_2c_4 - s_1s_4 & -c_1c_2s_4 - s_1c_4 & c_1s_2 & c_1d_3s_2 - s_1d_2 \\ s_1c_2c_4 + c_1s_4 & -s_1c_2s_4 + c_1c_4 & s_1s_2 & s_1d_3s_2 + c_1d_2 \\ -s_2c_4 & s_2c_4 & c_2 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.94} $$

$$ {}^0_5T = \begin{bmatrix} X & X & -c_1c_2s_4 - s_1c_4 & c_1d_3s_2 - s_1d_2 \\ X & X & -s_1c_2s_4 + c_1c_4 & s_1d_3s_2 + c_1d_2 \\ X & X & s_2s_4 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.95} $$

$$ {}^0_6T = \begin{bmatrix} X & X & c_1c_2c_4s_5 - s_1s_4s_5 + c_1s_2s_5 & c_1d_3s_2 - s_1d_2 \\ X & X & s_1c_2c_4s_5 + c_1s_4s_5 + s_1s_2s_5 & s_1d_3s_2 + c_1d_2 \\ X & X & -s_2c_4s_5 + c_5c_2 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.96} $$

As we can see the origin of frame $\{3\}$ is the same as the one for $\{4\}, \{5\}$ and $\{6\}$. All we need to keep for the computation of the orientation is just the third column of the transformation matrices. Now we can fill the 6×6 Jacobian in this case using the information above.

$$ J = \begin{pmatrix} \frac{\partial x_P}{\partial q_1} & \frac{\partial x_P}{\partial q_2} & \frac{\partial x_P}{\partial q_3} & 0 & 0 & 0 \\ {}^0\mathbf{z}_1 & {}^0\mathbf{z}_2 & 0 & {}^0\mathbf{z}_4 & {}^0\mathbf{z}_5 & {}^0\mathbf{z}_6 \end{pmatrix} \tag{4.97} $$

As we can see the 3×1 representation of the third orientation vector is 0 (since it is a prismatic link). Similarly, we easily fill the rest of the matrix as a function of ${}^0\mathbf{z}_1, {}^0\mathbf{z}_2, \ldots, {}^0\mathbf{z}_6$ and $\frac{\partial x_P}{\partial q_1}, \frac{\partial x_P}{\partial q_2}, \frac{\partial x_P}{\partial q_3}$. These expressions can be easily calculated using the transforms we calculated before. The Jacobian, J, is

$$\begin{bmatrix} -c_1 d_2 - s_1 s_2 d_3 & c_1 c_2 d_3 & c_1 s_2 & 0 & 0 & 0 \\ -s_1 d_2 + c_1 s_2 d_3 & s_1 c_2 d_3 & s_1 s_2 & 0 & 0 & 0 \\ 0 & -s_2 d_3 & c_2 & 0 & 0 & 0 \\ 0 & -s_1 & 0 & c_1 s_2 & -s_1 c_2 s_4 + c_1 c_4 & c_1 c_2 c_4 s_5 - s_1 s_4 s_5 + c_1 s_2 s_5 \\ 0 & c_1 & 0 & s_1 s_2 & -s_1 c_2 s_4 + c_1 c_4 & s_1 c_2 c_4 s_5 + c_1 s_4 s_5 + s_1 s_2 s_5 \\ 1 & 0 & 0 & c_2 & s_2 s_4 & -s_2 c_4 s_5 + c_5 c_2 \end{bmatrix}$$

Of course, all quantities in this matrix are expressed in frame $\{0\}$. Note that the vertical dimension of the basic Jacobian depends on the number of DOF of the mechanism, while the horizontal one is six (3 for the position and 3 for the orientation).

4.7.2 Jacobian in a Different Frame

As we mentioned above, we may want to express the Jacobian in different frames. The transformation matrix between two frames is

$$^B J = \begin{pmatrix} {}^B_A R & 0 \\ 0 & {}^B_A R \end{pmatrix} {}^A J \tag{4.98}$$

In practice the best frame to compute the Jacobian is in the middle of the chain because that makes the expressions of the elements of the Jacobian least complicated. Moving to frame $\{0\}$ can be done using the above transformation.

4.8 Kinematic Singularities

The work space of a manipulator generally contains a number of particular configurations that locally limits the end-effector mobility. Such configurations are called singular configurations. At a singular configuration, the end-effector locally loses the ability to move along or rotate about some direction in Cartesian space.

Note that these singularities are related to the kinematics of the manipulator and are obviously different from the singularities of the representation we have discussed earlier, which uniquely arise from the type of the selected representation.

For example the kinematic singularities for a 2 DOF revolute arm (see Figure 4.11) are the configurations where the two links are collinear. The end-effector cannot move along the common link direction.

Another example of singularity is the wrist singularity, which is common for the Stanford Scheinman Arm and the PUMA. This is the configuration when axis 4 and axis 6 are collinear, the end-effector cannot rotate about the normal to the plane defined by axes 4 and 5.

In such configurations, instantaneously the end-effector cannot rotate about that axis. In other words even though we can vary the joint velocities, the resulting linear or angular velocity at the end effector will be zero. Since the Jacobian is the mapping between these velocities, the analysis of kinematic singularities is closely connected to the Jacobian.

At a singular configuration, some columns of the Jacobian matrix become linearly dependent and the Jacobian losses rank. The phenomenon of singularity can then be studied by checking the determinant of the Jacobian, which is zero at singular configurations.

$$\det[J(\mathbf{q})] = 0 \tag{4.99}$$

Consider again the example of the 2 DOF revolute manipulator illustrated in Figure 4.11.

From simple geometric considerations we derive the coordinates of the end-effector point.

$$x = l_1 c_1 + l_2 c_{12} \tag{4.100}$$

$$y = l_1 s_1 + l_2 s_{12} \tag{4.101}$$

This leads to the Jacobian

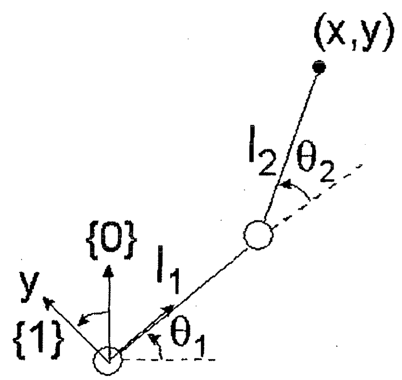

Figure 4.11: 2 DOF Example

$$^{0}J = \begin{pmatrix} -y & -l_2 s_{12} \\ x & l_2 c_{12} \end{pmatrix} \tag{4.102}$$

We now express the Jacobian in frame $\{1\}$ to further simplify its expression.

$$^{1}J = {}^{1}_{0}R\,{}^{0}J$$

with

$${}^{0}_{1}R = \begin{pmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{pmatrix} \tag{4.103}$$

Thus:

$$^{1}J = \begin{pmatrix} -l_2 s_2 & -l_2 s_2 \\ l_1 + l_2 c_2 & l_2 c_2 \end{pmatrix} \tag{4.104}$$

The above expressions show how the manipulator approaches a singularity as the angle θ_2 goes to zero. When $s_2 = 0$ the first row becomes $(0 \quad 0)$ and the rank of the Jacobian is 1.

$$^{1}J = \begin{pmatrix} 0 & 0 \\ l_1 + l_2 & l_2 \end{pmatrix} \tag{4.105}$$

Note that the determinant of the Jacobian does not depend on the frame where the matrix is defined.

$$\det[{}^{B}J] = \det \begin{bmatrix} {}^{B}_{A}R & 0 \\ 0 & {}^{B}_{A}R \end{bmatrix} \det[{}^{A}J] \tag{4.106}$$

Consider a small joint displacement $(\delta\theta_1, \delta\theta_2)$ from the singular configuration. The corresponding end-effector displacement $(\delta x, \delta y)$ expressed in frame $\{1\}$ is

$$\begin{pmatrix} {}^{1}\delta x \\ {}^{1}\delta y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ l_1 + l_2 & l_2 \end{pmatrix} \begin{pmatrix} \delta\theta_1 \\ \delta\theta_2 \end{pmatrix} \tag{4.107}$$

Thus:

$${}^{1}\delta x = 0 \tag{4.108}$$

and

$${}^{1}\delta y = (l_1 + l_2)\delta\theta_1 + l_2\delta\theta_2 \tag{4.109}$$

4.9 Jacobian at Wrist/End-Effector

The point at the end-effector, where linear and angular velocities are evaluated, varies with the robot's task, grasped object, or tools. Each selection of the end-effector point corresponds to a different Jacobian. The simplest Jacobian corresponds to the wrist point. The wrist point is fixed with respect to the end-effector and the Jacobian for any other point can be computed from the wrist Jacobian.

Consider a point E at the end-effector located with respect to the wrist point (origin of frame $\{n\}$) by a vector $\mathbf{p}_{we}$. The linear velocity, $\mathbf{v}_e$ at point E is

$$\mathbf{v}_e = \mathbf{v}_n + \omega_n \times \mathbf{p}_{ne} \tag{4.110}$$

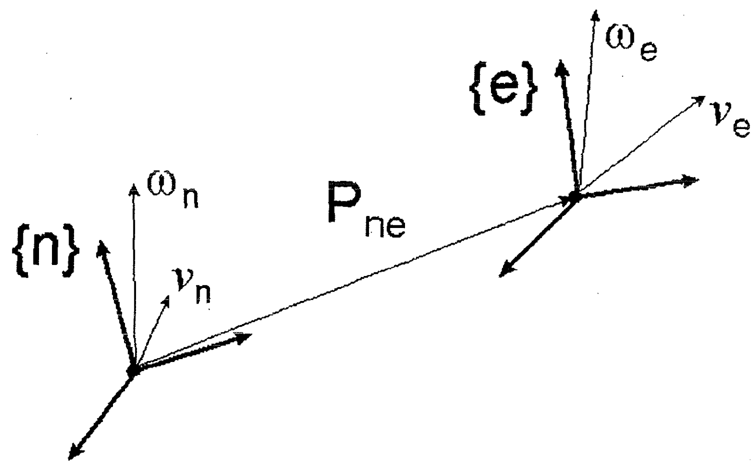

Figure 4.12: Jacobian at the End-effector

Since the angular velocity is the same at both points (E being fixed with respect to W), we have

$$\begin{cases} \mathbf{v}_e = \mathbf{v}_n - \mathbf{p}_{ne} \times \omega_n \\ \omega_e = \omega_n \end{cases} \tag{4.111}$$

Replacing $\mathbf{p}_{ne}\times$ by the cross product operator $\widehat{\mathbf{p}}_{ne}$ yields

$$^0J_e = \begin{pmatrix} I & -^0\widehat{\mathbf{p}}_{ne} \\ 0 & I \end{pmatrix} {^0J_n} \tag{4.112}$$

4.9.1 Example: 3 DOF RRR Arm

Let us consider the 3 DOF revolute planar mechanism shown in Figure 4.13.

The position coordinates of the end-effector wrist point are

$$x_W = l_1c_1 + l_2c_{12} \tag{4.113}$$
$$y_W = l_1s_1 + l_2s_{12} \tag{4.114}$$

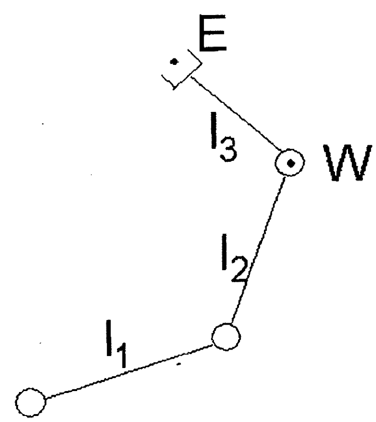

Figure 4.13: 3 DOF Example

at the end-effector point is:

$$x_E = l_1 c_1 + l_2 c_{12} + l_3 c_{123} \quad (4.115)$$
$$y_E = l_1 s_1 + l_2 s_{12} + l_3 s_{123} \quad (4.116)$$

The Jacobian in frame {0} for the wrist point is

$$J_W = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} & 0 \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad (4.117)$$

To get the Jacobian at the end-effector we will use vector $\mathbf{p}_{WE}$ in frame {0}:

$$^0J_E = \begin{pmatrix} I & -^0\widehat{\mathbf{p}}_{WE} \\ 0 & I \end{pmatrix} {}^0J_W \quad (4.118)$$

Thus the cross product operator is

$$
{}^0\mathbf{p}_{WE} = \begin{bmatrix} l_3c_{123} \\ l_3s_{123} \\ 0 \end{bmatrix} \Rightarrow {}^0\widehat{\mathbf{p}}_{WE} = \begin{pmatrix} 0 & 0 & l_3s_{123} \\ 0 & 0 & -l_3c_{123} \\ -l_3s_{123} & l_3c_{123} & 0 \end{pmatrix} \tag{4.119}
$$

The Jacobian at point E is

$$
{}^0J_E = \begin{pmatrix} I & -{}^0\widehat{\mathbf{p}}_{WE} \\ 0 & I \end{pmatrix} \begin{pmatrix} {}^0J_{v(W)} \\ {}^0J_{\omega(W)} \end{pmatrix} = \begin{pmatrix} {}^0J_{v(W)} - {}^0\widehat{\mathbf{p}}_{WE}{}^0J_{\omega(W)} \\ {}^0J_{\omega(W)} \end{pmatrix} \tag{4.120}
$$

Here:

$$
-{}^0\widehat{\mathbf{p}}_{WE}{}^0J_{\omega(W)} = \begin{pmatrix} -l_3s_{123} & -l_3s_{123} & -l_3s_{123} \\ l_3c_{123} & l_3c_{123} & l_3c_{123} \\ 0 & 0 & 0 \end{pmatrix} \tag{4.121}
$$

If we perform the multiplications we obtain the following 3 columns for the Jacobian associated with the linear velocity and the overall Jacobian follows..

$$
{}^0J_E = \begin{bmatrix} -l_1s_1 - l_2s_{12} - l_3s_{123} & -l_2s_{12} - l_3s_{123} & -l_3s_{123} \\ l_1c_1 + l_2c_{12} + l_3c_{123} & l_2c_{12} + l_3c_{123} & l_3c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \tag{4.122}
$$

We can verify these results using the time derivatives of (x, y, z) as before.

4.10 Static Forces

Another application of the Jacobian is to define the relationship between forces applied at the end-effector and torques needed at the joints to support these forces. We described the relationship between linear velocities and angular velocities at the end-effector and the joint velocities. Here we consider the relationship between end-effector forces and moments as related to joint torques. We will denote by **f** and **n** the static force and moment applied by the end-effector to the environment. $\tau_1, \tau_2, \ldots, \tau_n$ are the torques needed at the joints of the manipulator to produce **f** and **n**.

4.10.1 Force Propagation

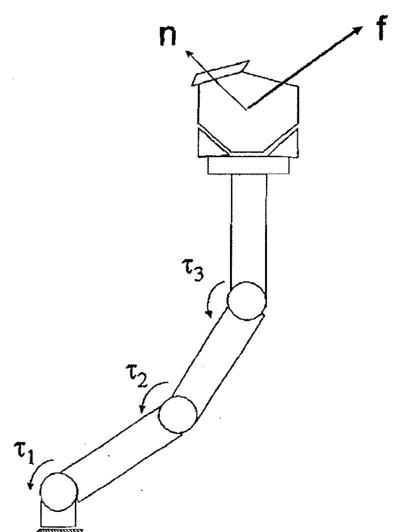

Figure 4.14: Static Forces

One way to establish this relationship is through propagation of the forces along the kinematic chain, similar to the velocity propagation from link to link. In fact as we will see later in considering the dynamics of the manipulator, velocities are propagated up the kinematic chain after which forces are propagated back in the opposite direction. As we propagate we can eliminate internal forces that are supported by the structure. This is done by projecting all forces at the joints. To analyze the static forces, let us imagine that we isolate the links of the manipulator into components, which can be treated as separate rigid bodies.

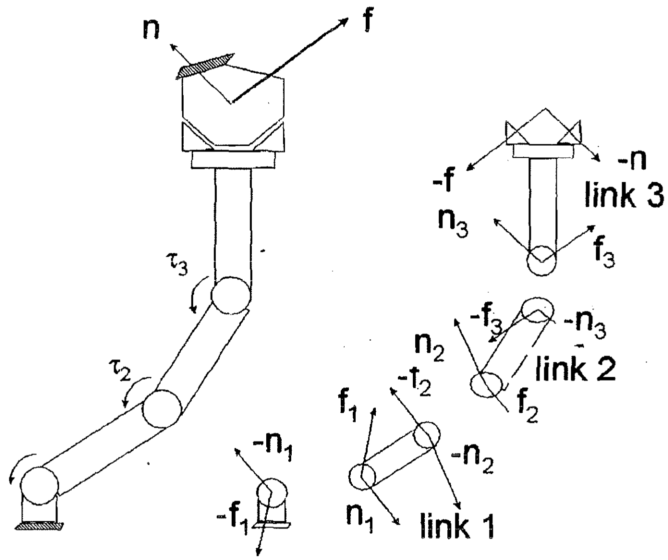

Figure 4.15: Force and Moments Cancellation

For each rigid body, we will consider all forces and moments that act on it and will then set the conditions for bring it to equilibrium. Let us consider the rigid body i (link i). In order for this rigid body to be at equilibrium, the sum of all forces and moments with respect to any point on the rigid body must be equal to zero. For link i, we have

$$\mathbf{f}_i + (-\mathbf{f}_{i+1}) = 0 \tag{4.123}$$

We select the origin of frame $\{i\}$ for the moment computation. This leads to the equation

$$\mathbf{n}_i + (-\mathbf{n}_{i+1}) + \mathbf{p}_{i+1} \times (-\mathbf{f}_{i+1}) = 0 \tag{4.124}$$

The above two relationships can be written recursively as follows:

$$\mathbf{f}_i = \mathbf{f}_{i+1} \tag{4.125}$$

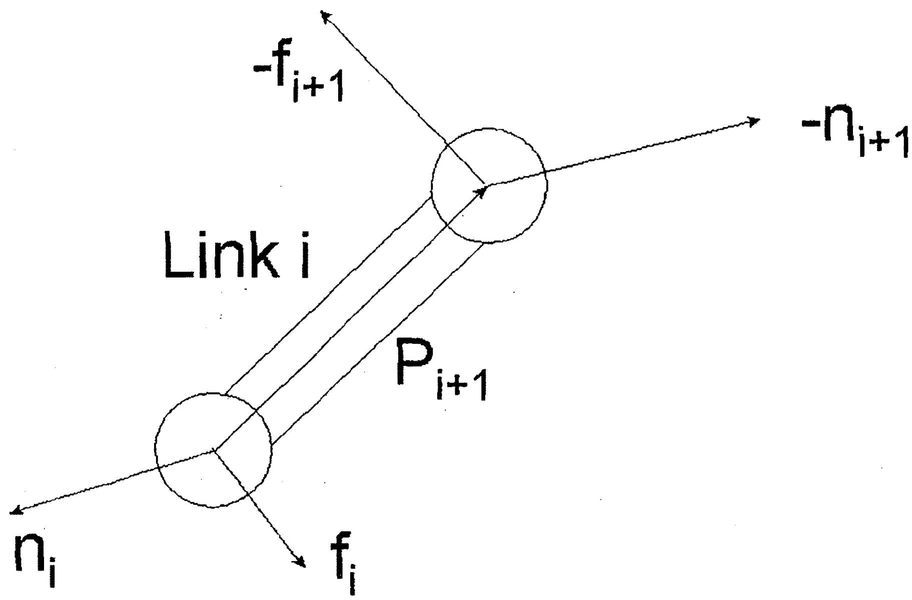

Figure 4.16: Link Equilibrium

$$\mathbf{n}_i = \mathbf{n}_{i+1} + \mathbf{p}_{i+1} \times \mathbf{f}_{i+1} \tag{4.126}$$

We need to guarantee that we eliminate in these equations the components that will be transmitted to the ground through the structure of the mechanism. We will do that by projecting the equations along the joint axis and propagating these relationships along the kinematic chain from the end-effector to the ground. These relationships are as follows: For a prismatic joint $\tau_i = \mathbf{f}_i^T \mathbf{z}_i$ and for a revolute joint $\tau_i = \mathbf{n}_i^T \mathbf{z}_i$.

For link n

$$^n\mathbf{f}_n = {}^n f \tag{4.127}$$

$$^n\mathbf{n}_n = {}^n n + {}^n\mathbf{p}_{n+1} \times {}^n f \tag{4.128}$$

and for link i

$$ {}^{i}\mathbf{f}_i = {}^{i}_{i+1}R^{i+1}\mathbf{f}_{i+1} \tag{4.129} $$

$$ {}^{i}\mathbf{n}_i = {}^{i}_{i+1}R^{i+1}\mathbf{n}_{i+1} + {}^{i}\mathbf{p}_{i+1} \times {}^{i}\mathbf{f}_i \tag{4.130} $$

This iterative process leads to a linear relationship between end-effector forces and moments and joint torques. The analysis of this relationship shows that it is simply the transpose of the Jacobian matrix.

$$ \tau = J^T\mathbf{F} $$

where $\mathbf{F}$ is the vector combining end-effector forces and moments. The above relationship is the dual of the relationship we have established earlier between the end-effector linear angular velocities and joint velocities.

Earlier we derived similar equations for propagating angular and linear velocities along the links. These equations are

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}R_i {}^{i}\omega_i + \dot{\theta}_{i+1} {}^{i+1}\mathbf{z}_{i+1} \tag{4.131} $$

$$ {}^{i+1}\mathbf{v}_{i+1} = {}^{i+1}R_{i+1}({}^{i}\mathbf{v}_i + {}^{i}\omega_i \times {}^{i}\mathbf{p}_{i+1}) + \dot{d}_{i+1} {}^{i+1}\mathbf{z}_{i+1} \tag{4.132} $$

Starting from the first fixed link, we can propagate to find the velocities at the end-effector, and then extract the Jacobian matrix.

4.10.2 Example: 3 DOF RRR Arm

Let us illustrate this method on the 3 DOF revolute manipulator we have been using in this Chapter.

For the linear velocity we obtain:

$$ \mathbf{v}_{P_1} = 0 \tag{4.133} $$

$$ \mathbf{v}_{P_2} = \mathbf{v}_{P_1} + \omega_1 \times P_2 \tag{4.134} $$

$$ \mathbf{v}_{P_3} = \mathbf{v}_{P_2} + \omega_2 \times P_3 \tag{4.135} $$

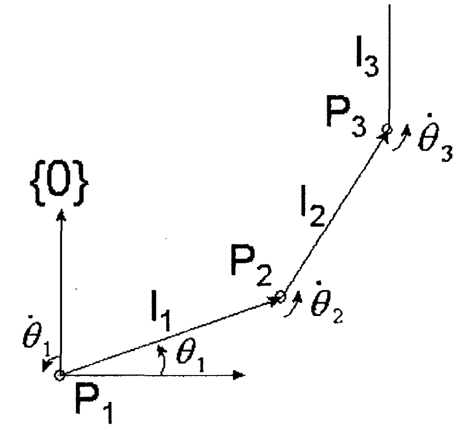

Figure 4.17: Example of Velocity Propagation

or

$$^0\mathbf{v}_{P_2} = 0 + \begin{bmatrix} 0 & -\dot{\theta}_1 & 0 \\ \dot{\theta}_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} l_1 c_1 \\ l_1 s_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -l_1 s_1 \\ l_1 c_1 \\ 0 \end{bmatrix} \dot{\theta}_1 \tag{4.136}$$

and

$$^0\mathbf{v}_{P_3} = \begin{bmatrix} -l_1 s_1 \\ l_1 c_1 \\ 0 \end{bmatrix} \dot{\theta}_1 + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} (\dot{\theta}_1 + \dot{\theta}_2)\,^0\mathbf{p}_3 \tag{4.137}$$

$$^0\mathbf{v}_{P_3} = \begin{bmatrix} -l_1 s_1 \\ l_1 c_1 \\ 0 \end{bmatrix} \dot{\theta}_1 + \begin{bmatrix} -l_2 s_{12} \\ l_2 c_{12} \\ 0 \end{bmatrix} (\dot{\theta}_1 + \dot{\theta}_2) \tag{4.138}$$

The angular velocities are simple since they are all rotations about the Z-axis perpendicular to the plane of the paper.

$$^0\omega_3 = (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)\,^0\mathbf{z}_0 \tag{4.139}$$

In matrix form

$$ {}^0\mathbf{v}_{P_3} = \begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} & 0 \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \tag{4.140} $$

and

$$ {}^0\omega_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \tag{4.141} $$

from which we obtain the Jacobian:

$$ \begin{pmatrix} \mathbf{v} \\ \omega \end{pmatrix} = J \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} \tag{4.142} $$

This Jacobian is clearly the same as previously calculated using the explicit method.

4.10.3 Virtual Work

The relationship between end-effector forces and joint torques can be directly established using the *virtual work principle.* This principal states that *at static equilibrium the virtual work of all applied forces is equal to zero.*

The virtual work principal allows to avoid computing and eliminating internal forces. Since internal forces do not produce any work, they are not involved in the analysis.

Joint torques and end-effector forces are the only applied or active forces for this mechanism. Let $\mathbf{F}$ be the vector of applied forces and moments at the end-effector,

$$ \mathbf{F} = \begin{pmatrix} \mathbf{f} \\ \mathbf{n} \end{pmatrix} \tag{4.143} $$

At static equilibrium, the virtual work is

$$\tau^T \delta \mathbf{q} + (-\mathbf{F})^T \delta \mathbf{x} = 0 \tag{4.144}$$

Note that the minus sign is due to the fact that forces at the end effector are applied by the environment to the end-effector.

Using the relationship

$$\delta \mathbf{x} = J \delta \mathbf{q}$$

yields

$$\tau = J^T \mathbf{F} \tag{4.145}$$

This is an important relationship not only for the the analysis of static forces but also for robot control.

4.11 More on Explicit Form: J_v

We have seen how the linear motion Jacobian, J_v, can be obtained from direct differentiations of the end-effector position vector. We develop here the explicit form for obtaining this matrix. The expression for the linear velocity was found in the form

$$\mathbf{v} = \sum_{i=1}^{n} [\epsilon_i \mathbf{z}_i + \bar{\epsilon}_i (\mathbf{z}_i \times \mathbf{p}_{in})] \dot{q}_i \tag{4.146}$$

$$\mathbf{v} = [\epsilon_1 \mathbf{z}_1 + \bar{\epsilon}_1 (\mathbf{z}_1 \times \mathbf{p}_{1n})] \dot{q}_1 + [\epsilon_2 \mathbf{z}_2 + \bar{\epsilon}_2 (\mathbf{z}_2 \times \mathbf{p}_{2n})] \dot{q}_2 + \cdots + [\epsilon_n \mathbf{z}_n + \bar{\epsilon}_n (\mathbf{z}_n \times \mathbf{p}_{nn})] \dot{q}_n \tag{4.147}$$

and the corresponding Jacobian is

$$J_v = \left[\epsilon_1 \mathbf{z}_1 + \bar{\epsilon}_1 (\mathbf{z}_1 \times \mathbf{p}_{1n}) \quad \cdots \quad \epsilon_n \mathbf{z}_n + \bar{\epsilon}_n (\mathbf{z}_n \times \mathbf{p}_{nn}) \right] \dot{\mathbf{q}} \tag{4.148}$$

In this form, J_v is expressed in terms of the $\mathbf{z}_i$ vectors and $\mathbf{p}_{in}$ vectors associated with the various links. Combining the linear and angular parts, the Jacobian J is

$$\begin{pmatrix} (\epsilon_1\mathbf{z}_1 + \bar{\epsilon}_1\mathbf{z}_1 \times \mathbf{p}_{1n}) & \cdots & (\epsilon_{n-1}\mathbf{z}_{n-1} + \bar{\epsilon}_{n-1}\mathbf{z}_{n-1} \times \mathbf{p}_{(n-1)n}) & \epsilon_n\mathbf{z}_n \\ \bar{\epsilon}_1\mathbf{z}_1 & \cdots & \bar{\epsilon}_{n-1}\mathbf{z}_{n-1} & \bar{\epsilon}_n\mathbf{z}_n \end{pmatrix}. \tag{4.149}$$

To express this matrix in a given frame, all vectors should be evaluated in that frame. The cross product $(\mathbf{z}_i \times \mathbf{p}_{in})$ can be evaluated in frame $\{i\}$. Again, since the components in $\{i\}$ of $\mathbf{z}_i$ are independent of frame $\{i\}$, we define

$$\widehat{Z} =^i \widehat{\mathbf{z}}_i = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The components in frame $\{i\}$ of cross product vectors $(\mathbf{z}_i \times \mathbf{p}_{in})$ are simply $(\widehat{Z}\ {}^i\mathbf{p}_{in})$.

The components in the frame $\{i\}$ of the vector $\mathbf{p}_{in}$ are given in the last column of the transformation ${}^i_nT$.

The expression in frame $\{0\}$ of the Jacobian matrix, ${}^0J$ is given by

$$\begin{pmatrix} {}^0_1R(\epsilon_1 Z + \bar{\epsilon}_1\widehat{Z}^1\mathbf{p}_{1n}) & \cdots & {}^0_{n-1}R(\epsilon_{n-1}Z + \bar{\epsilon}_{n-1}\widehat{Z}^{n-1}\mathbf{p}_{(n-1)n}) & {}^0_nR\epsilon_n\mathbf{Z} \\ {}^0_1R\bar{\epsilon}_1 Z & \cdots & {}^0_{n-1}R\bar{\epsilon}_{n-1}Z & {}^0_nR\bar{\epsilon}_n Z \end{pmatrix}. \tag{4.150}$$

4.11.1 Stanford Scheinman Arm Example

Applying the explicit form of J_v to the Stanford Scheinman arm, we can easily (by setting the numerical values of ϵ_i) write the Jacobian as

$${}^0J = \begin{pmatrix} {}^0(\mathbf{z}_1 \times \mathbf{p}_{13}) & {}^0(\mathbf{z}_2 \times \mathbf{p}_{23}) & {}^0\mathbf{z}_3 & 0 & 0 & 0 \\ {}^0\mathbf{z}_1 & {}^0\mathbf{z}_2 & 0 & {}^0\mathbf{z}_4 & {}^0\mathbf{z}_5 & {}^0\mathbf{z}_6 \end{pmatrix} \tag{4.151}$$

$\mathbf{p}_{13}$ is given in ${}^1_3T$ in frame $\{1\}$

$$ {}^1_3T = \begin{pmatrix} & {}^1_3R & & {}^1\mathbf{p}_{13} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_2 & 0 & s_2 & d_3s_2 \\ 0 & 1 & 0 & d_2 \\ -s_2 & 0 & c_2 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.152} $$

To express $(\mathbf{z}_1 \times \mathbf{p}_{13})$ in frame $\{0\}$, we have

$$ {}^0(\mathbf{z}_1 \times \mathbf{p}_{13}) = {}^0_1R.({}^1\mathbf{z}_1 \times {}^1\mathbf{p}_{13}) \tag{4.153} $$

The computation of ${}^i\mathbf{z}_i \times {}^iP_{in}$ in frame $\{i\}$ is simply

$$ ({}^i\mathbf{z}_i \times {}^i\mathbf{p}) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{p}_x \\ \mathbf{p}_y \\ \mathbf{p}_z \end{pmatrix} = \begin{pmatrix} -\mathbf{p}_y \\ \mathbf{p}_x \\ 0 \end{pmatrix} \tag{4.154} $$

For ${}^1\mathbf{z}_1 \times {}^iP_{13}$, this computation is

$$ {}^1\mathbf{p}_{13} = \begin{pmatrix} d_3s_2 \\ d_2 \\ d_3c_2 \end{pmatrix} \tag{4.155} $$

and

$$ {}^1\mathbf{z}_1 \times {}^1\mathbf{p}_{13} = \begin{pmatrix} -d_2 \\ d_3s_2 \\ 0 \end{pmatrix} \tag{4.156} $$

In frame $\{0\}$ this is

$$ {}^0_1R({}^1\mathbf{z}_1 \times {}^1\mathbf{p}_{13}) = \begin{pmatrix} c_1 & s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -d_2 \\ d_3s_2 \\ 0 \end{pmatrix} = \begin{pmatrix} -c_1d_2 - s_1s_2d_3 \\ -s_1d_2 + c_1s_2d_3 \\ 0 \end{pmatrix} \tag{4.157} $$

For ${}^0(\mathbf{z}_2 \times \mathbf{p}_{23})$, we can similarly obtain

$$ {}^2\mathbf{p}_{23} = \begin{bmatrix} 0 \\ -d_3 \\ 0 \end{bmatrix} \leftarrow {}^2_3T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -d_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.158} $$

and

$$ {}^2\mathbf{z}_2 \times {}^2\mathbf{p}_{23} = \begin{pmatrix} d_3 \\ 0 \\ 0 \end{pmatrix} \tag{4.159} $$

Since

$$ {}^0_2R = \begin{pmatrix} c_1c_2 & X & X \\ s_1c_2 & X & X \\ -s_2 & X & X \end{pmatrix} \tag{4.160} $$

we obtain

$$ {}^0_2R({}^2\mathbf{z}_2 \times {}^2\mathbf{p}_{23}) = \begin{pmatrix} c_1c_2d_3 \\ s_1c_2d_3 \\ -s_2d_3 \end{pmatrix} \tag{4.161} $$

Finally $\mathbf{z}_3$ in frame $\{0\}$ is

$$ {}^0\mathbf{z}_3 = \leftarrow {}^0_3R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{pmatrix} c_1s_2 \\ s_1s_2 \\ c_2 \end{pmatrix} \tag{4.162} $$

The Jacobian in frame $\{0\}$ is, as expected, the same as the one derived earlier:

$$\begin{bmatrix} -c_1d_2 - s_1s_2d_3 & c_1c_2d_3 & c_1s_2 & 0 & 0 & 0 \\ -s_1d_2 + c_1s_2d_3 & s_1c_2d_3 & s_1s_2 & 0 & 0 & 0 \\ 0 & -s_2d_3 & c_2 & 0 & 0 & 0 \\ 0 & -s_1 & 0 & c_1s_2 & -s_1c_2s_4 + c_1c_4 & c_1c_2c_4s_5 - s_1s_4s_5 + c_1s_2s_5 \\ 0 & c_1 & 0 & s_1s_2 & -s_1c_2s_4 + c_1c_4 & s_1c_2c_4s_5 + c_1s_4s_5 + s_1s_2s_5 \\ 1 & 0 & 0 & c_2 & s_2s_4 & -s_2c_4s_5 + c_5c_2 \end{bmatrix}$$

There is yet another approach to compute the vectors $\mathbf{p}_{in}$, this is discussed in the next section.

4.11.2 $\mathbf{p}_{in}$ Derivation

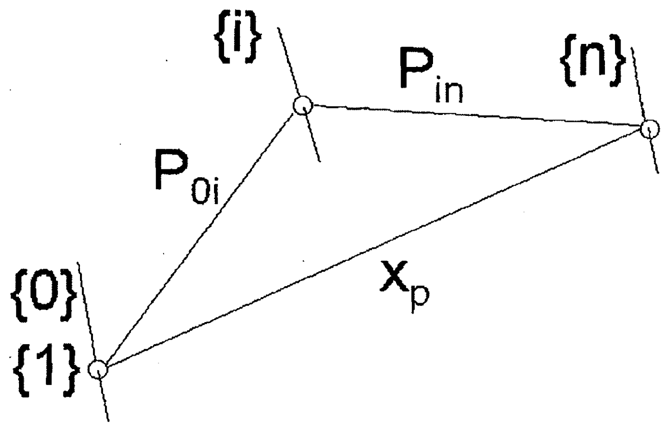

Figure 4.18: Computing $\mathbf{p}_{in}$

The computation in frame $\{i\}$ of the vector $\mathbf{p}_{in}$ requires ${}^i_nT$. However, this transformation is often not explicitly available, as only the matrices: ${}^0_1T$, ${}^0_2T$, ..., ${}^0_iT$, ..., ${}^0_nT$ are computed. In this case, it is more efficient to express $\mathbf{p}_{in}$ as

$$\mathbf{p}_{in} = \mathbf{x}_p - \mathbf{p}_{0i}.$$

The vector $\mathbf{x}_p$ and $\mathbf{p}_{0i}$ are expressed in frame $\{0\}$, ${}^0\mathbf{p}_{0i}$ is given in ${}^0_iT$.

The computation of $({}^0_iR\ \widehat{Z}\ {}^i\mathbf{p}_{in})$ that appears in (4.150) can then be replaced by

$$({}^0_iR\ \widehat{Z}\ {}^i\mathbf{p}_{in}) \quad \Longrightarrow \quad ({}^0_iR\ \widehat{Z}\ {}^0_iR^T)\ ({}^0\mathbf{x}_p - {}^0\mathbf{p}_{0i}).$$