Position & Orientation & State

"At home with Homogenous Transformations"

METR 4202: Advanced Control & **Robotics**
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Lecture # 2
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Continuing from last week…

- Robotics Definition:

"A robot is a reprogrammable, multifunctional manipulator designed to move material, parts, tools, or specialized devices through variable programmed motions for the performance of a variety of tasks."
(Robotics Institute of America)

It is a machine which has some ability to interact with physical objects and to be given electronic programming to do a specific task or to do a whole range of tasks or actions.
(Wikipedia)

Programmable electro-mechanical systems that adapt to identify and leverage a **structural characteristic** of the environment
(Surya)

Types of Robotics Systems

- Manipulators
- Multiple
- Mobile

Enabling Mathematics:

- Computational Kinematics
- Operational Space

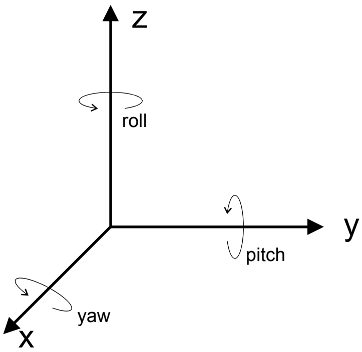
- Behaviour based "Reflexive" control rules

-Probabilistic methods

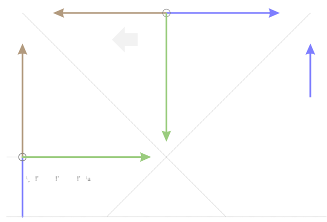
Schedule

Week	Date	Lecture (F: 9-10:30, 42-212)
1	26-Jul	Introduction
2	**2-Aug**	**Representing Position & Orientation & State (Frames, Transformation Matrices & Affine Transformations)**
3	9-Aug	Robot Kinematics and Dynamics
4	16-Aug	Robot Dynamics & Control
5	23-Aug	Sensors & Measurement
6	30-Aug	Perception
7	6-Sep	Computer Vision & Localization (SFM/SLAM)
8	13-Sep	Localization and Navigation
9	20-Sep	State-Space Modelling
	27-Sep	State-Space Control
10	4-Oct	*Study break*
11	11-Oct	Motion Planning
12	18-Oct	Vision-based control (+ Prof. P. Corke or + Prof. M. Srinivasan)
13	25-Oct	Applications in Industry (+ Prof. S. LaValle) & Course Review

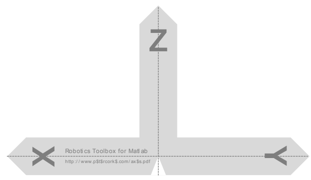
Today's Lecture is about: Frames & Their Mathematics



- Make one (online):
  - SpnS Template

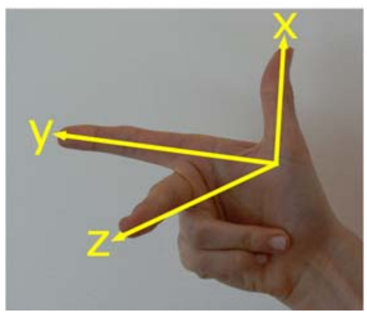


  - Peter Corke's template



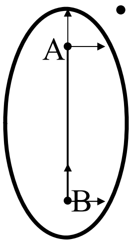
Don't Confuse a Frame with a Point

- Points
  - Position Only – Doesn't Encode Orientation
- Frame
  - Encodes both position and orientation
  - Has a "handedness"

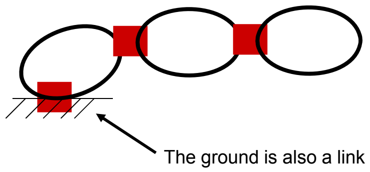


Kinematics Definition

- **<u>Kinematics</u>**: The study of motion in space (without regard to the forces which cause it)



- Assume:
  - Points with ***right-hand <u>Frames</u>***
  - *Rigid-bodies* in 3D-space (6-dof)
  - 1-dof joints: Rotary (R) or Prismatic (P) (5 constraints)

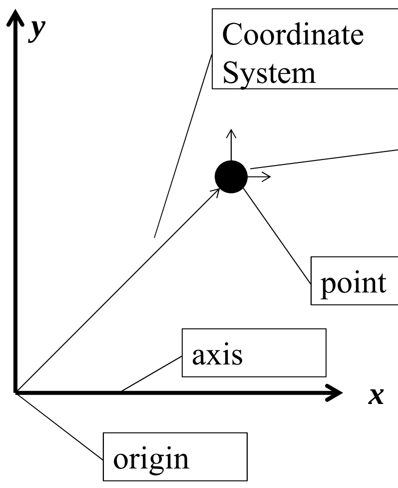


N links
M joints
→DOF = 6N-5M
→ If N=M, then DOF=N.

Kinematics

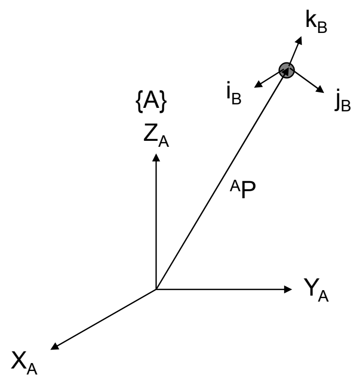
- Kinematic modelling is one of the most important analytical tools of robotics.
- Used for modelling mechanisms, actuators and sensors
- Used for on-line control and off-line programming and simulation
- In mobile robots kinematic models are used for:
  - steering (control, simulation)
  - perception (image formation)
  - sensor head and communication antenna pointing
  - world modelling (maps, object models)
  - terrain following (control feedforward)
  - gait control of legged vehicles

Basic Terminology



Coordinate System

- The position and orientation as specified only make sense with respect to some coordinate system



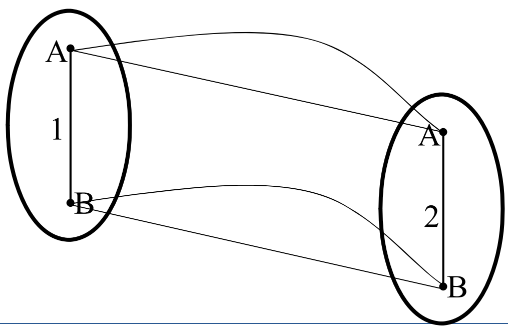
Frames of Reference

- A frame of reference defines a coordinate system relative to some point in space
- It can be specified by a position and orientation relative to other frames
- The *inertial frame* is taken to be a point that is assumed to be fixed in space

- Two types of motion:
  - Translation
  - Rotation

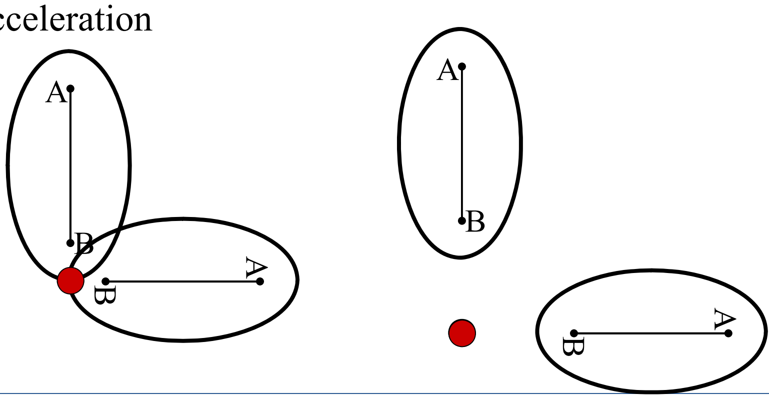
Translation

- A motion in which a straight line with in the body keeps the same direction during the
  - **Rectilinear Translation:** Along straight lines
  - **Curvilinear Translation**: Along curved lines



Rotation

- The particles forming the rigid body move in parallel planes along circles centered around the same fixed axis (called the **axis of rotation**).
- Points on the axis of rotation have zero velocity and acceleration



Rotation: Representations

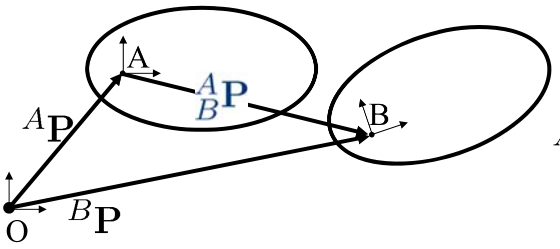
- Orientation are not "Cartesian"
  - Non-commutative
  - Multiple representations

- Some representations:
  - **Rotation Matrices**: Homegenous Coordinates
  - Euler Angles: 3-sets of rotations in sequence
  - Quaternions: a 4-paramameter representation that exploits ½ angle properties
  - Screw-vectors (from Charles Theorem) : a canonical representation, its reciprocal is a "wrench" (forces)

Position and Orientation [1]

- A **position** vectors specifies the location of a **point** in 3D (Cartesian) space

$$\mathbf{P} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$



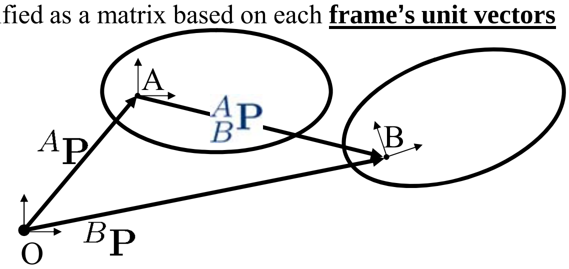
$$^{A}\mathbf{P} + {}^{A}\mathbf{P}^{B} - {}^{B}\mathbf{P} = 0$$

$$^{A}\mathbf{P}^{B} = {}^{A}\mathbf{P}_{B} = {}^{A}_{B}\mathbf{P} = \begin{bmatrix} ^{B}p_x \\ ^{B}p_y \\ ^{B}p_z \end{bmatrix} - \begin{bmatrix} ^{A}p_x \\ ^{A}p_y \\ ^{A}p_z \end{bmatrix}$$

- BUT we **also** concerned with its orientation in 3D space.
  This is specified as a matrix based on each **frame's unit vectors**

Position and Orientation [2]

- Orientation in 3D space:
  This is specified as a matrix based on each **frame's unit vectors**



- Describes {B} relative to {A}
  → The orientation of frame {B} relative to coordinate frame {A}
- Written "from {A} to {B}" or "given {A} getting to {B}"

$$^{A}\mathbf{R}_{B} = {}^{A}_{B}\mathbf{R} = \begin{bmatrix} ^{A}\hat{\imath}_{B} & ^{A}\hat{\jmath}_{B} & ^{A}\hat{k}_{B} \end{bmatrix}$$

- **Columns** are **{B} written in {A}**

Position and Orientation [3]

- The rotations can be analysed based on the unit components …
- That is: the components of the orientation matrix are the unit vectors projected **onto** the unit directions of the reference frame

$$ {}^{A}_{B}\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

$$ \begin{array}{c|ccc} {}^{A}_{B}R & (b_x)\,\hat{\imath}_B & (b_y)\,\hat{\jmath}_B & (b_z)\,\hat{k}_B \\ \hline (a_x)\,\hat{\imath}_A & \hat{\imath}_B \cdot \hat{\imath}_A & \hat{\jmath}_B \cdot \hat{\imath}_A & \hat{k}_B \cdot \hat{\imath}_A \\ (a_y)\,\hat{\jmath}_A & \hat{\imath}_B \cdot \hat{\jmath}_A & \hat{\jmath}_B \cdot \hat{\jmath}_A & \hat{k}_B \cdot \hat{\jmath}_A \\ (a_z)\,\hat{k}_A & \hat{\imath}_B \cdot \hat{k}_A & \hat{\jmath}_B \cdot \hat{k}_A & \hat{k}_B \cdot \hat{k}_A \end{array} $$

Position and Orientation [4]

- Rotation is orthonormal

$$ \begin{array}{c|ccc} {}^{A}_{B}R & (b_x)\,\hat{\imath}_B & (b_y)\,\hat{\jmath}_B & (b_z)\,\hat{k}_B \\ \hline (a_x)\,\hat{\imath}_A & \hat{\imath}_B \cdot \hat{\imath}_A & \hat{\jmath}_B \cdot \hat{\imath}_A & \hat{k}_B \cdot \hat{\imath}_A \\ (a_y)\,\hat{\jmath}_A & \hat{\imath}_B \cdot \hat{\jmath}_A & \hat{\jmath}_B \cdot \hat{\jmath}_A & \hat{k}_B \cdot \hat{\jmath}_A \\ (a_z)\,\hat{k}_A & \hat{\imath}_B \cdot \hat{k}_A & \hat{\jmath}_B \cdot \hat{k}_A & \hat{k}_B \cdot \hat{k}_A \end{array} $$

- The of a rotation matrix inverse = the transpose

$$ \mathbf{R} \cdot \mathbf{R}^{T} = 1 $$

→ thus, the ***rows*** are **{A} written in {B}**

$$ {}^{B}_{A}\mathbf{R} = {}^{A}_{B}\mathbf{R}^{T} = {}^{A}_{B}\mathbf{R}^{-1} $$

Position and Orientation [5]: A note on orientations

- Orientations, as defined earlier, are represented by three orthonormal vectors
- Only three of these values are unique and we often wish to define a particular rotation using three values (it's easier than specifying 9 orthonormal values)
- There isn't a unique method of specifying the angles that define these transformations

Position and Orientation [7]

- Shortcut Notation:

$$\cos(\theta_a) = c\theta_a = \mathbf{c_a}$$
$$\sin(\theta_a) = s\theta_a = \mathbf{s_a}$$

$$\cos(\theta_a + \theta_b) = \mathbf{c_{ab}}$$

$$\therefore \mathbf{s_{ab}} = \boxed{\quad ?}$$

Position and Orientation [8]

- Rotation Formula about the 3 Principal Axes by θ

X:
$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Y:
$$\mathbf{R}_y = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

Z:
$$\mathbf{R}_z = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Euler Angles

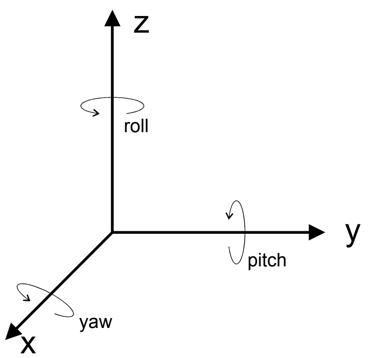
- Minimal representation of orientation (α,β,γ)
- Represent a rotation about an axis of a **moving** coordinate frame
  → ${}^{A}_{B}\mathbf{R}$: Moving frame **B** w/r/t fixed A
- The location of the axis of each successive rotation depends on the previous one! …
- So, Order Matters (12 combinations, why?)
- Often Z-Y-X:
  - α: rotation about the **z** axis
  - β: rotation about the rotated **y** axis
  - γ: rotation about the twice rotated **x** axis
- Has singularities! … (e.g., β=±90°)

Fixed Angles

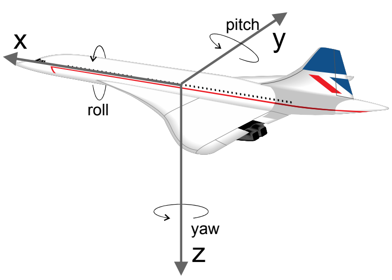
- Represent a rotation about an axis of a **fixed** coordinate frame.
- Again 12 different orders
- Interestingly:
  3 rotations about 3 axes of a **fixed** frame define the same orientation as the same 3 rotations taken in the **opposite order** of the **moving** frame
- For X-Y-Z:
  - ψ: rotation about $\mathbf{x}_A$ (sometimes called "yaw")
  - θ: rotation about $\mathbf{y}_A$ (sometimes called "pitch")
  - φ: rotation about $\mathbf{z}_A$ (sometimes called "roll")

Roll – Pitch – Yaw

- In many Kinematics References:



- In many Engineering Applications:



→ Be careful:
This name is given to other conventions too!

Euler Angles [1]: **X-Y-Z Fixed Angles** (Roll-Pitch-Yaw)

- One method of describing the orientation of a Frame {B} is:
  - Start with the frame coincident with a known reference {A}. Rotate {B} first about X_A by an angle γ, then about Y_A by an angle β and finally about Z_A by an angle α.

$$
\begin{aligned}
{}^{A}R_{BXYZ}(\gamma,\beta,\alpha) &= R_Z(\alpha)R_Y(\beta)R_X(\gamma) \\
&= \begin{bmatrix} c_\alpha & -s_\alpha & 0 \\ s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\beta & 0 & s_\beta \\ 0 & 1 & 0 \\ -s_\beta & 0 & c_\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\gamma & -s_\gamma \\ 0 & s_\gamma & c_\gamma \end{bmatrix} \\
&= \begin{bmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}
\end{aligned}
$$

Euler Angles [2]: Z-Y-X Euler Angles

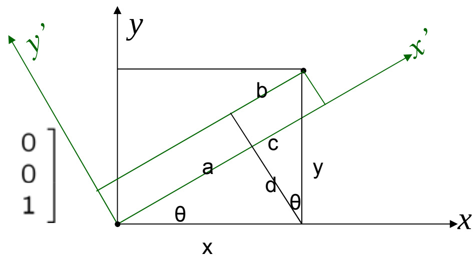
- Another method of describing the orientation of {B} is:
  - Start with the frame coincident with a known reference {A}. Rotate {B} first about Z_B by an angle α, then about Y_B by an angle β and finally about X_B by an angle γ.

$$
\begin{aligned}
{}^{A}R_{BZ'Y'X'}(\gamma,\beta,\alpha) &= R_Z(\alpha)R_Y(\beta)R_X(\gamma) \\
&= \begin{bmatrix} c_\alpha & -s_\alpha & 0 \\ s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\beta & 0 & s_\beta \\ 0 & 1 & 0 \\ -s_\beta & 0 & c_\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\gamma & -s_\gamma \\ 0 & s_\gamma & c_\gamma \end{bmatrix} \\
&= \begin{bmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}
\end{aligned}
$$

Position and Orientation [6]:
"Proof" of Principal Rotation Matrix Terms

- Geometric:

$$\mathbf{R}_z = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$a = x\cos\theta,\ b = y\sin\theta$

$c = y\cos\theta,\ d = x\sin\theta$

Thus:

$x' = x\cos\theta + y\sin\theta$

$y' = -x\sin\theta + y\cos\theta$

Unit Quaternion (ϵ_0, ϵ_1, ϵ_2, ϵ_3) [1]

- Does not suffer from singularities

$$\epsilon \equiv \epsilon_0 + \left(\epsilon_1\hat{\mathbf{i}} + \epsilon_2\hat{\mathbf{j}} + \epsilon_3\hat{\mathbf{k}}\right)$$

- Uses a "4-number" to represent orientation

$$ii = jj = kk = -1$$
$$ij = k, jk = i, ki = j, ji = -k, kj = -1, ik = -j$$

- Product:

$$\begin{aligned} \mathbf{ab} &= (a_0b_0 - a_1b_1 - a_2b_2 + a_3b_3) \\ &\quad + (a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2)\,\hat{i} \\ &\quad + (a_0b_2 + a_2b_0 + a_3b_1 + a_1b_3)\,\hat{j} \\ &\quad + (a_0b_3 + a_3b_0 + a_1b_2 - a_2b_1)\,\hat{k} \end{aligned}$$

- Conjugate:

$$\tilde{\epsilon} \equiv \epsilon_0 - \epsilon_1\hat{\mathbf{i}} - \epsilon_2\hat{\mathbf{j}} - \epsilon_3\hat{\mathbf{k}}$$
$$\epsilon\tilde{\epsilon} = \tilde{\epsilon}\epsilon = \epsilon_0^2 + \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2$$

Unit Quaternion [2]: Describing Orientation

- Set $\epsilon_0 = 0$
  Then $\mathbf{p}=(p_x,p_y,p_z)$ → $\mathbf{p} = p_x\hat{\mathbf{i}} + p_y\hat{\mathbf{j}} + p_z\hat{\mathbf{k}}$

- Then given ϵ
  the operation $\epsilon \mathbf{p} \tilde{\epsilon}$: rotates **p** about $(\epsilon_1, \epsilon_2, \epsilon_3)$

- Unit Quaternion → Rotation Matrix

$$\mathbf{R} = \begin{pmatrix} 1-2\left(\epsilon_2^2+\epsilon_3^2\right) & 2\left(\epsilon_1\epsilon_2-\epsilon_0\epsilon_3\right) & 2\left(\epsilon_1\epsilon_3-\epsilon_0\epsilon_2\right) \\ 2\left(\epsilon_1\epsilon_2-\epsilon_0\epsilon_3\right) & 1-2\left(\epsilon_1^2+\epsilon_3^2\right) & 2\left(\epsilon_2\epsilon_3-\epsilon_0\epsilon_1\right) \\ 2\left(\epsilon_1\epsilon_3-\epsilon_0\epsilon_2\right) & 2\left(\epsilon_2\epsilon_3-\epsilon_0\epsilon_1\right) & 1-2\left(\epsilon_1^2+\epsilon_2^2\right) \end{pmatrix}$$

Direction Cosine

- Uses the Direction Cosines (read dot products) of the Coordinate Axes of the moving frame with respect to the fixed frame

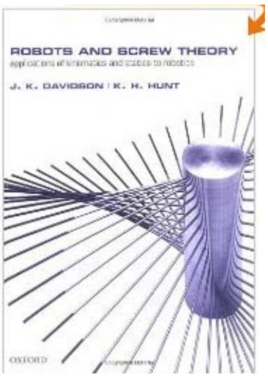
$$^{A}\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$$
$$^{A}\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$$
$$^{A}\mathbf{w} = w_x\mathbf{i} + w_y\mathbf{j} + w_z\mathbf{k}$$

- It forms a rotation matrix!

${}^{A}_{B}R$	$(b_x)\,\hat{\imath}_B$	$(b_y)\,\hat{\jmath}_B$	$(b_z)\,\hat{k}_B$
$(a_x)\,\hat{\imath}_A$	$\hat{\imath}_B\cdot\hat{\imath}_A$	$\hat{\jmath}_B\cdot\hat{\imath}_A$	$\hat{k}_B\cdot\hat{\imath}_A$
$(a_y)\,\hat{\jmath}_A$	$\hat{\imath}_B\cdot\hat{\jmath}_A$	$\hat{\jmath}_B\cdot\hat{\jmath}_A$	$\hat{k}_B\cdot\hat{\jmath}_A$
$(a_z)\,\hat{k}_A$	$\hat{\imath}_B\cdot\hat{k}_A$	$\hat{\jmath}_B\cdot\hat{k}_A$	$\hat{k}_B\cdot\hat{k}_A$

Screw Displacements

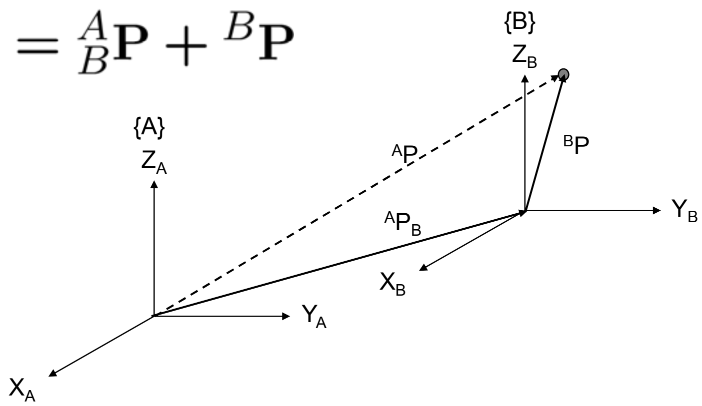
- Comes from the notion that all motion can be viewed as a rotation (Rodrigues formula)
- Define a vector along the axis of motion (screw vector)
  - Rotation (screw angle)
  - Translation (pitch)
  - Summations → via the screw triangle!



Coordinate Transformations [1]

- Translation Again:
  If {B} is translated with respect to {A} **without rotation**, then it is a vector sum

$$ {}^{A}\mathbf{P} = {}^{A}_{B}\mathbf{P} + {}^{B}\mathbf{P} $$

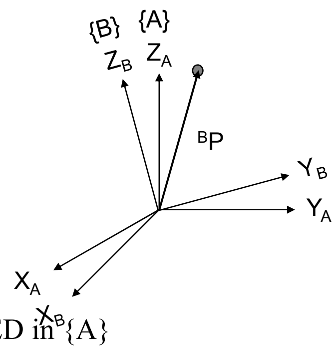


Coordinate Transformations [2]

- Rotation Again:
  {B} is rotated with respect to {A} then
  use rotation matrix to determine new components

- NOTE: $${}^{A}\mathbf{P} = {}^{A}_{B}\mathbf{R}\,{}^{B}\mathbf{P}$$
  - The Rotation matrix's ***subscript*** matches the position vector's **superscript**

$${}^{A}\mathbf{P} = {}^{A}_{[\![B]\!]}\mathbf{R}^{[\![B]\!]}\mathbf{P}$$

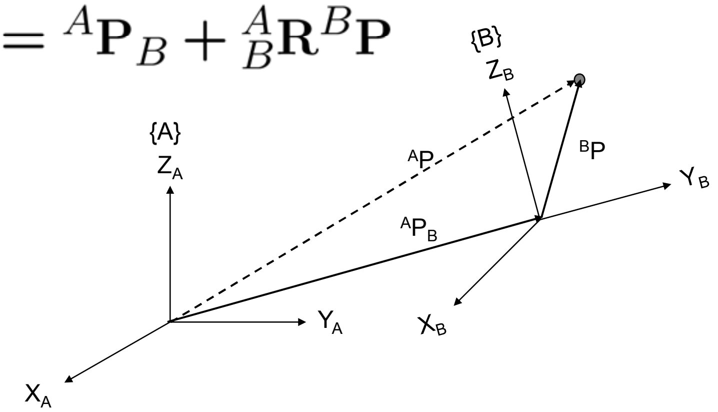


  - This gives Point Positions of {B} ORIENTED in {A}

Coordinate Transformations [3]

- Composite transformation:
  {B} is moved with respect to {A}:

$${}^{A}\mathbf{P} = {}^{A}\mathbf{P}_{B} + {}^{A}_{B}\mathbf{R}\,{}^{B}\mathbf{P}$$



General Coordinate Transformations [1]

- A compact representation of the translation and rotation is known as the **Homogeneous Transformation**

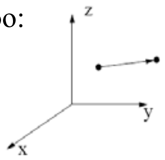
$$ {}^{A}_{B}\mathbf{T} = \begin{bmatrix} & {}^{A}_{B}\mathbf{R} & & {}^{A}\mathbf{P}_{B} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

- This allows us to cast the rotation and translation of the general transform in a single matrix form

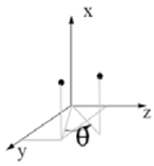
$$ \begin{bmatrix} {}^{A}\mathbf{P} \\ 1 \end{bmatrix} = {}^{A}_{B}\mathbf{T} \begin{bmatrix} {}^{B}\mathbf{P} \\ 1 \end{bmatrix} $$

General Coordinate Transformations [2]

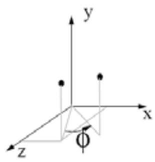
- Similarly, fundamental orthonormal transformations can be represented in this form too:



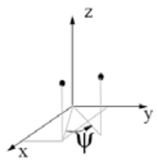
$$ Trans(u, v, w) = \begin{bmatrix} 1 & 0 & 0 & u \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & w \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



$$ Rotx(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\theta & -s\theta & 0 \\ 0 & s\theta & c\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



$$ Roty(\phi) = \begin{bmatrix} c\phi & 0 & s\phi & 0 \\ 0 & 1 & 0 & 0 \\ -s\phi & 0 & c\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

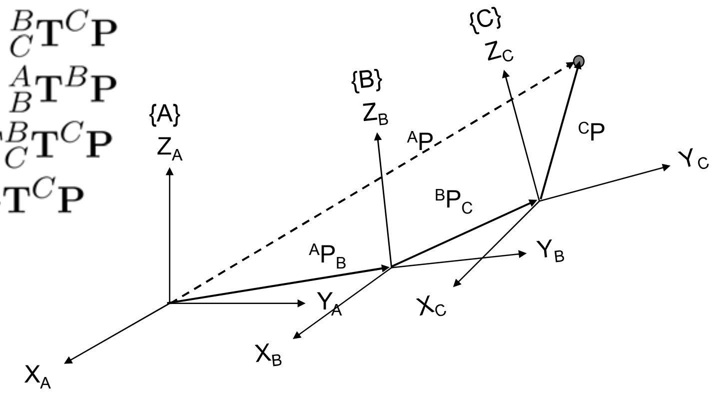


$$ Rotz(\psi) = \begin{bmatrix} c\psi & -s\psi & 0 & 0 \\ s\psi & c\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

General Coordinate Transformations [3]

- Multiple transformations compounded as a chain

$$
\begin{aligned}
{}^{B}\mathbf{P} &= {}^{B}_{C}\mathbf{T}\,{}^{C}\mathbf{P} \\
{}^{A}\mathbf{P} &= {}^{A}_{B}\mathbf{T}\,{}^{B}\mathbf{P} \\
&= {}^{A}_{B}\mathbf{T}\,{}^{B}_{C}\mathbf{T}\,{}^{C}\mathbf{P} \\
&= {}^{A}_{C}\mathbf{T}\,{}^{C}\mathbf{P}
\end{aligned}
$$



$$
{}^{A}_{C}\mathbf{T} = \begin{bmatrix} {}^{A}_{B}\mathbf{R}\,{}^{B}_{C}\mathbf{R} & {}^{A}\mathbf{P}_{B} + {}^{A}_{B}\mathbf{R}\,{}^{B}\mathbf{P}_{C} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}
$$

Homogenous Coordinates

$$
\hat{p} = \begin{bmatrix} \rho p_x & \rho p_y & \rho p_z & \rho \end{bmatrix}^T
$$

- ρ is a scaling value

Homogenous Transformation

$$\begin{bmatrix} {}^{A}R_{B} & {}^{A}p \\ \gamma & \rho \end{bmatrix}$$

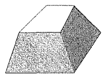
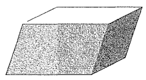
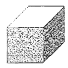
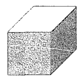
- γ is a projective transformation

Projective Transformations …

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $\mathbf{l}_\infty$.
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, **I**, **J** (see section 2.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area

p.44, R. Hartley and A. Zisserman. *Multiple View Geometry in Computer Vision*

Projective Transformations & Other Transformations of 3D Space

Group	Matrix	Distortion	Invariant properties
Projective 15 dof	$\begin{bmatrix} \mathrm{A} & \mathbf{t} \\ \mathbf{v}^{\mathsf{T}} & v \end{bmatrix}$		Intersection and tangency of surfaces in contact. Sign of Gaussian curvature.
Affine 12 dof	$\begin{bmatrix} \mathrm{A} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$		Parallelism of planes, volume ratios, centroids. The plane at infinity, π_∞, (see section 3.5).
Similarity 7 dof	$\begin{bmatrix} s\mathrm{R} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$		The absolute conic, Ω_∞, (see section 3.6).
Euclidean 6 dof	$\begin{bmatrix} \mathrm{R} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$		Volume.

p.78, R. Hartley and A. Zisserman. *Multiple View Geometry in Computer Vision*

Special Orthogonal & Special Euclidean

- SO(n): R**o**tations

$$SO(n) = \{R \in \mathbb{R}^{n\times n} : RR^T = I, \det R = +1\}.$$

$$\exp(\widehat{\omega}\theta) = e^{\widehat{\omega}\theta} = I + \theta\widehat{\omega} + \frac{\theta^2}{2!}\widehat{\omega}^2 + \frac{\theta^3}{3!}\widehat{\omega}^3 + \dots$$

- SE(n): Transformations of EUCLIDEAN space

$$SE(n) := \mathbb{R}^n \times SO(n).$$

$$SE(3) = \{(p, R) : p \in \mathbb{R}^3, R \in SO(3)\} = \mathbb{R}^3 \times SO(3).$$

Forward Kinematics [1]

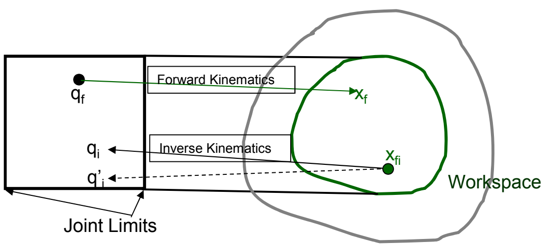
- Forward kinematics is the process of chaining homogeneous transforms together. For example to:
  - Find the articulations of a mechanism, or
  - the fixed transformation between two frames which is known in terms of linear and rotary parameters.
- Calculates the final position from the **machine** (**joint variables**)
- Unique for an open kinematic chain (**serial arm**)
- "Complicated" (multiple solutions, etc.) for a closed kinematic chain (**parallel arm**)

Forward Kinematics [2]

- Can think of this as "spaces":
  - Operation space (x,y,z,α,β,γ): The robot's position & orientation
  - Joint space ($\theta_1 \ldots \theta_n$): A state-space vector of joint variables

$$\vec{\mathbf{x}} = \begin{bmatrix} \vec{\mathbf{p}} \\ \vec{\mathbf{\Theta}} \end{bmatrix}$$

$$\vec{\mathbf{q}} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$



Summary

- Many ways to view a rotation
  - Rotation matrix
  - Euler angles
  - Quaternions
  - Direction Cosines
  - Screw Vectors

- Homogenous transformations
  - Based on homogeneous coordinates

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