State Space Control

A First-Order Means of Control!

METR 4202: Advanced Control & **Robotics**

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Lecture # 10

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Schedule

Week	Date	Lecture (F: 9-10:30, 42-212)
1	26-Jul	Introduction
2	2-Aug	Representing Position & Orientation & State (Frames, Transformation Matrices & Affine Transformations)
3	9-Aug	Robot Kinematics
4	16-Aug	Robot Dynamics & Control
5	23-Aug	Robot Trajectories & Motion
6	30-Aug	Sensors & Measurement
7	6-Sep	Perception / Computer Vision
8	13-Sep	Localization and Navigation
9	20-Sep	State-Space Modelling
10	**27-Sep**	**State-Space Control**
	4-Oct	*Study break*
11	11-Oct	Motion Planning
12	18-Oct	Vision-based control (+ Prof. P. Corke or Prof. M. Srinivasan)
13	25-Oct	Applications in Industry (+ Prof. S. LaValle) & Course Review

Announcements: We're Working On It!

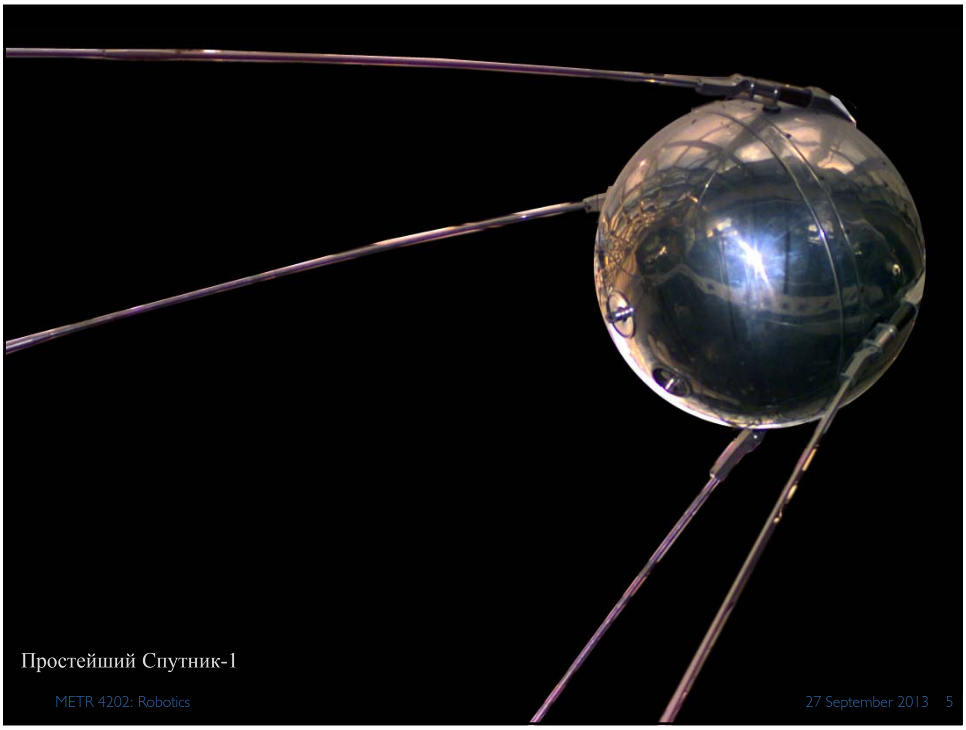
- **Grades:**
  - I am still working on it
  - I've read the course profile 1100× ☺
- **Lab 3:**
  - Working on it!
- Cool Robotics Share Site
  - Jared is making a "blog". URL Soon!
  - He is still working on it !! ☺

Cool Robotics Video Share

Welcome to

State-Space!

(Why the big type?)



State-Space!

государственный контроль пространства

- *It's Russian for Control*

- **Dynamic systems** are described as differential equations (as compared to transfer functions)

- **Stability** is approached via the theory of Liapunov instead of frequency-domain methods (Bode and Nyquist)

- **Optimisation of System Performance** via calculus of variations (Pontryagin) (as compared to Wiener-Hopf methods)

State-Space Control

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x}$$

(That can not be all of it? There has to be more to it than this…)

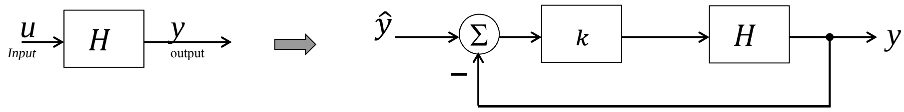
State-Space Control

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{u}$$

Benefits:

- Characterises the process by systems of coupled, first-order differential equations
- More general mathematical model
  - MIMO, time-varying, nonlinear
- Mathematically esoteric (who needs practical solutions)
- Yet, well suited for digital computer implementation
  - That is: based on vectors/matrices (think LAPACK → MATLAB)

Difference Equations & Feedback



- Start with the Open-Loop:

$$y = Hu$$

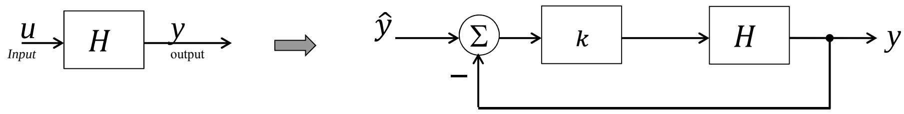
- Close the loop:

$$u = ke = k(\hat{y} - y) \rightarrow y = H[k(\hat{y} - y)]$$

$$\rightarrow \mathrm{y} = \frac{Hk}{1+Hk}\hat{y}$$

- All easy! (yesa!)

Difference Equations & Feedback



- Now add delay (image the plant is a replica with a delay τ)

$$y(t) = u(t - \tau)$$

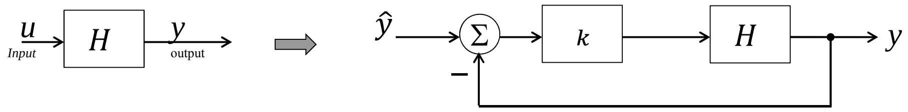
- Close the loop:

$$u(t-\tau) = ke(t-\tau) = k\,[\hat{y}(t-\tau) - \mathrm{y}(t-\tau)]$$

$$\rightarrow \mathrm{y(t)} = k\,[\hat{y}(t-\tau) - \mathrm{y}(t-\tau)]$$

- Notice we have a difference equation!

Difference Equations & Feedback



- What happens with a single delay and a unit step?

$$u(t) = k \text{ for } 0<t<\tau$$

$$y(t) = u(t-\tau) \text{ for } \tau<t<2\tau$$

- Then with feedback we get:

$$u(t) = k(1-k) = k - k^2$$

$$y(t) = k - k^2 + k^3 + \cdots + (-1)^{n-1}k^{n-1}$$

- If k<1: then:

$$\rightarrow \lim y(t) = \frac{k}{1+k}$$

Controllability

Controllability matrix

- If you can write it in CCF, then the system equations must be linearly independent.
- Transformation by any nonsingular matrix preserves the controllability of the system.
- Thus, a nonsingular controllability matrix means $\boldsymbol{x}$ can be driven to any value.

State evolution

- Consider the system matrix relation:

$$\dot{\boldsymbol{x}} = \mathbf{F}\boldsymbol{x} + \mathbf{G}u$$
$$y = \mathbf{H}\boldsymbol{x} + Ju$$

The time solution of this system is:

$$\boldsymbol{x}(t) = e^{\mathbf{F}(t-t_0)}\, \boldsymbol{x}(t_0) + \int_{t_0}^{t} = e^{\mathbf{F}(t-\tau)}\, \mathbf{G}u(\tau)d\tau$$

If you didn't know, the matrix exponential is:

$$e^{\mathbf{K}t} = \mathbf{I} + \mathbf{K}t + \frac{1}{2!}\mathbf{K}^2t^2 + \frac{1}{3!}\mathbf{K}^3t^3 + \cdots$$

Stability

- We can solve for the natural response to initial conditions $\boldsymbol{x_0}$:

$$\boldsymbol{x}(t) = e^{p_i t}\boldsymbol{x}_0$$
$$\therefore \dot{\boldsymbol{x}}(t) = p_i e^{p_i t}\boldsymbol{x}_0 = \mathbf{F}e^{p_i t}\boldsymbol{x}_0$$

Clearly, a system will be stable provided
$\mathrm{eig}(\mathbf{F}) < 0$

Characteristic polynomial

- From this, we can see $\mathbf{F}\boldsymbol{x}_0 = p_i\boldsymbol{x}_0$

or, $(p_i\mathrm{I} - \mathbf{F})\boldsymbol{x}_0 = 0$

which is true only when $\det(p_i\mathrm{I} - \mathbf{F})\boldsymbol{x}_0 = 0$

Aka. the characteristic equation!

- We can reconstruct the CP in s by writing:

$$\det(s\mathrm{I} - \mathbf{F})\boldsymbol{x}_0 = 0$$

Great, so how about control?

- Given $\dot{\boldsymbol{x}} = \mathbf{F}\boldsymbol{x} + \mathbf{G}u$, if we know $\mathbf{F}$ and $\mathbf{G}$, we can design a controller $u = -\mathbf{K}\boldsymbol{x}$ such that

$$\mathrm{eig}(\mathbf{F} - \mathbf{GK}) < 0$$

- In fact, if we have full measurement and control of the states of $\boldsymbol{x}$, we can position the poles of the system in arbitrary locations!

(Of course, that never happens in reality.)

Example: PID control

- Consider a system parameterised by three states:
  - x_1, x_2, x_3
  - where $x_2 = \dot{x}_1$ and $x_3 = \dot{x}_2$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -2 \end{bmatrix} \boldsymbol{x} - \mathbf{K}u$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \boldsymbol{x} + 0u$$

x_2 is the output state of the system;

x_1 is the value of the integral;

x_3 is the velocity.

- We can choose **K** to move the eigenvalues of the system as desired:

$$\det\begin{bmatrix} 1-K_1 & & \\ & 1-K_2 & \\ & & -2-K_3 \end{bmatrix} = \mathbf{0}$$

All of these eigenvalues must be positive.

It's straightforward to see how adding derivative gain K_3 can stabilise the system.

Just scratching the surface

- There is a lot of stuff to state-space control

- One lecture (or even two) can't possibly cover it all in depth

Go play with Matlab and check it out!

Discretisation FTW!

- We can use the time-domain representation to produce difference equations!

$$\boldsymbol{x}(kT+T) = e^{\mathbf{F}T}\,\boldsymbol{x}(kT) + \int_{kT}^{kT+T} e^{\mathbf{F}(kT+T-\tau)}\mathbf{G}u(\tau)d\tau$$

Notice $\boldsymbol{u}(\tau)$ is not based on a discrete ZOH input, but rather an integrated time-series.

We can structure this by using the form:

$$u(\tau) = u(kT), \qquad kT \le \tau \le kT+T$$

Discretisation FTW!

- Put this in the form of a new variable:

$$\eta = kT + T - \tau$$

Then:

$$\boldsymbol{x}(kT+T) = e^{\boldsymbol{F}T}\boldsymbol{x}(kT) + \left(\int_{kT}^{kT+T} e^{\boldsymbol{F}\eta}d\eta\right)\boldsymbol{G}u(kT)$$

Let's rename $\boldsymbol{\Phi} = e^{\boldsymbol{F}T}$ and $\boldsymbol{\Gamma} = \left(\int_{kT}^{kT+T} e^{\boldsymbol{F}\eta}d\eta\right)\boldsymbol{G}$

Discrete state matrices

So,

$$\boldsymbol{x}(k+1) = \boldsymbol{\Phi}\boldsymbol{x}(k) + \boldsymbol{\Gamma}u(k)$$
$$y(k) = \mathbf{H}\boldsymbol{x}(k) + \mathbf{J}\boldsymbol{u}(k)$$

Again, $\boldsymbol{x}(k+1)$ is shorthand for $\boldsymbol{x}(kT+T)$

Note that we can also write $\boldsymbol{\Phi}$ as:

$$\boldsymbol{\Phi} = \mathbf{I} + \mathbf{F}T\boldsymbol{\Psi}$$

where

$$\boldsymbol{\Psi} = \mathbf{I} + \frac{\mathbf{F}T}{2!} + \frac{\mathbf{F}^2T^2}{3!} + \cdots$$

Simplifying calculation

- We can also use $\boldsymbol{\Psi}$ to calculate $\boldsymbol{\Gamma}$
  - Note that:

$$\Gamma = \sum_{k=0}^{\infty} \frac{\mathbf{F}^k T^k}{(k+1)!} T\mathbf{G}$$
$$= \boldsymbol{\Psi} T\mathbf{G}$$

$\boldsymbol{\Psi}$ itself can be evaluated with the series:

$$\boldsymbol{\Psi} \cong \mathbf{I} + \frac{\mathbf{F}T}{2}\left\{\mathbf{I} + \frac{\mathbf{F}T}{3}\left[\mathbf{I} + \cdots \frac{\mathbf{F}T}{n-1}\left(\mathbf{I} + \frac{\mathbf{F}T}{n}\right)\right]\right\}$$

State-space z-transform

We can apply the z-transform to our system:

$$(z\mathbf{I} - \mathbf{\Phi})\boldsymbol{X}(z) = \mathbf{\Gamma} U(k)$$
$$Y(z) = \mathbf{H}\boldsymbol{X}(z)$$

which yields the transfer function:

$$\frac{Y(z)}{\boldsymbol{X}(z)} = G(z) = \mathbf{H}(z\mathbf{I} - \mathbf{\Phi})^{-1}\mathbf{\Gamma}$$

State-space control design

- Design for discrete state-space systems is just like the continuous case.
  - Apply linear state-variable feedback:

$$u = -\mathbf{K}\boldsymbol{x}$$

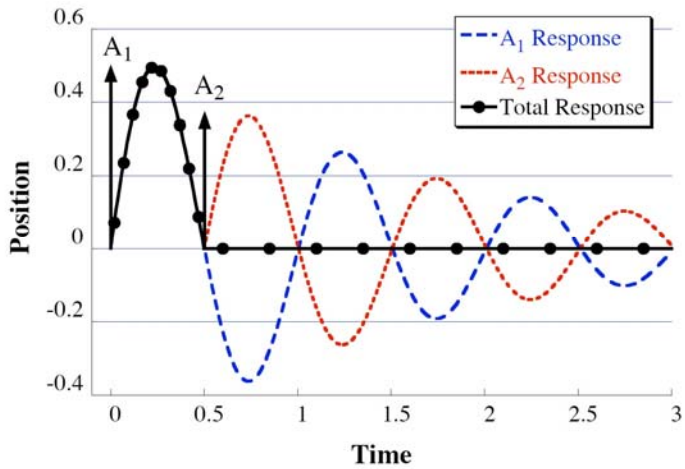
such that $\det(z\mathbf{I} - \mathbf{\Phi} + \mathbf{\Gamma}\mathbf{K}) = \alpha_c(z)$

where $\alpha_c(z)$ is the desired control characteristic equation

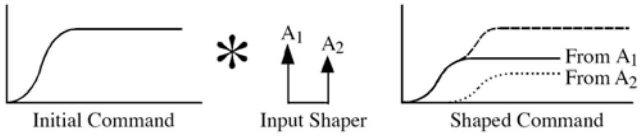
Predictably, this requires the system controllability matrix $\boldsymbol{\mathcal{C}} = [\mathbf{\Gamma} \quad \mathbf{\Phi\Gamma} \quad \mathbf{\Phi}^2\mathbf{\Gamma} \quad \cdots \quad \mathbf{\Phi}^{n-1}\mathbf{\Gamma}]$ to be full-rank.

Example: Command Shaping

Command Shaping



Command Shaping



- Zero Vibration (ZV)

$$\begin{bmatrix} A_i \\ t_i \end{bmatrix} = \begin{bmatrix} \dfrac{1}{1+K} & \dfrac{K}{1+K} \\ 0 & \dfrac{T_d}{2} \end{bmatrix} \qquad K = e^{\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)}$$

- Zero Vibration and Derivative (ZVD)

$$\begin{bmatrix} A_i \\ t_i \end{bmatrix} = \begin{bmatrix} \dfrac{1}{(1+K)^2} & \dfrac{2K}{(1+K)^2} & \dfrac{K^2}{(1+K)^2} \\ 0 & \dfrac{T_d}{2} & T_d \end{bmatrix}$$

Example II: Estimation

Along multiple dimensions

State Space

- We collect our set of uncertain variables into a vector …
  $\mathbf{x} = [x_1, x_2, \ldots, x_N]^T$

- The set of values that **x** might take on is termed the *state space*

- There is a *single* true value for **x**,
  but it is unknown

State Space Dynamics

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$$H(s) = C(sI - A)^{-1}B$$

Measured versus True

- Measurement errors are inevitable
- So, add Noise to State...
  - State Dynamics becomes:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + w$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} + v$$

- Can represent this as a "Normal" Distribution

$$\mathcal{N}(x;\mu,\sigma) = \frac{1}{\left(\sqrt{2\pi}\right)\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Recovering The Truth

- Numerous methods
- Termed "Estimation" because we are trying to estimate the truth from the signal
- A strategy discovered by Gauss
- Least Squares in Matrix Representation

$$\begin{bmatrix} p_0 \\ p_1 \end{bmatrix} = \begin{bmatrix} n & \sum_1^n t_i \\ \sum_1^n t_i & \sum_1^n t_i^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_1^n z_i \\ \sum_1^n t_i z_i \end{bmatrix}$$

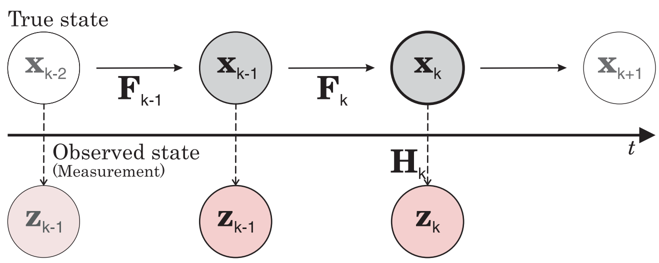
Recovering the Truth: Terminology

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{u} + \mathbf{w}$$

$$\mathbf{z} = \mathbf{H}\mathbf{x} + \mathbf{v}$$

$\mathbf{x}$: the state vector
$\mathbf{x}_{A|B}$: the state of $\mathbf{x}$ at time A based on data taken up to time B
$\hat{\mathbf{x}}$: estimate of the true state vector
$\mathbf{F}$: system dynamics matrix in continuous time (equivalent to $\mathbf{A}$ in Eq. 1)
$\mathbf{G}$: system control matrix relating deterministic input, $\mathbf{u}$, to the state (equivalent to $\mathbf{B}$ in Eq. 1)
$\mathbf{H}$: measurement matrix in continuous time (equivalent to $\mathbf{C}$ in Eq. 2)
$\mathbf{F}_i$: system model in **discrete** time at $t = t_i$
$\mathbf{H}_i$: measurement model in **discrete** time at $t = t_i$
$\mathbf{P}_i$: estimate covariance in **discrete** time at $t = t_i$
$\mathbf{w}$: process uncertainty (noise) vector (of type $\mathcal{N}(0, s)$)
$\mathbf{Q}$: process noise matrix, $\mathbf{Q} = E\left[ww^\mathsf{T}\right]$
$\mathbf{Q}_i$: $\mathbf{Q}$ in discrete time at $t = t_i$
$\mathbf{v}$: measurement noise vectors (of type $\mathcal{N}(0, \sigma)$)
$\mathbf{R}_i$: the measurement variance matrix, $\mathbf{R} = E\left[vv^\mathsf{T}\right]$, in discrete time at $t = t_i$

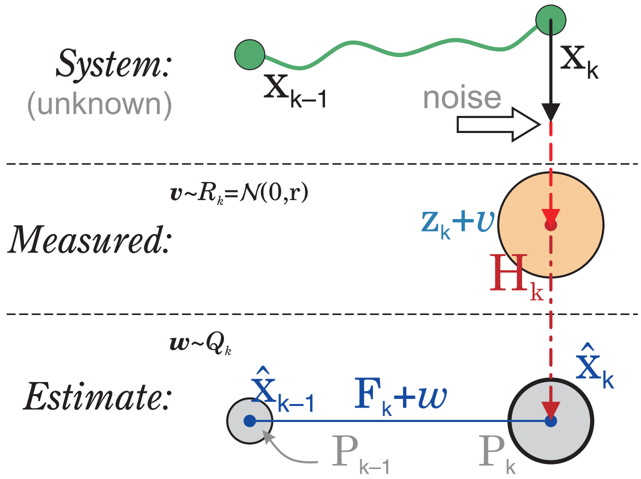
General Problem…



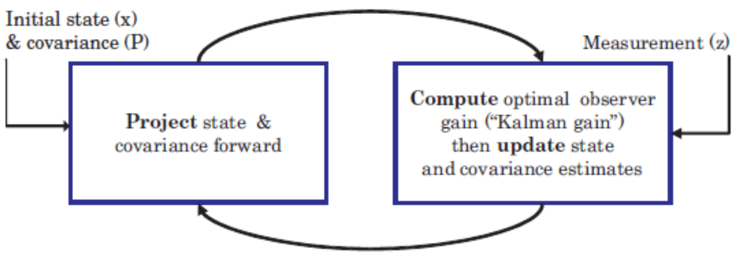
Duals and Dual Terminology

	Estimation		Control
Model:	$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x}$ (discrete: $\mathbf{x} = \mathbf{F}_k\mathbf{x}$)	$\leftrightarrow$	$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, $\mathbf{A} = \mathbf{F}^T$
Regulates:	P (covariance)	$\leftrightarrow$	M (performance matrix)
Minimized function:	Q (or GQG^T)	$\leftrightarrow$	V
Optimal Gain:	K	$\leftrightarrow$	G
Completeness law:	Observability	$\leftrightarrow$	Controllability

Estimation Process in Pictures



Kalman Filter Process



KF Process in Equations

$$\begin{aligned}
\text{Prediction: } \hat{\mathbf{x}}_{k|k-1} &= \mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1|k-1}, && \text{(state prediction)}\\
\mathbf{P}_{k|k-1} &= \mathbf{Q}_{k-1} + \mathbf{F}_{k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}^{\mathsf{T}}{}_{k-1}, && \text{(covariance prediction)}\\
\text{Kalman Gain: } \mathbf{K}_k &= \mathbf{P}_{k|k-1}\mathbf{H}^{\mathsf{T}}[\mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^{\mathsf{T}} + \mathbf{R}_k]^{-1}, &&\\
\text{Update: } \mathbf{P}_{k|k} &= [\mathbf{I} - \mathbf{K}_k\mathbf{H}]\mathbf{P}_{k|k-1}, && \text{(covariance update)}\\
\hat{\mathbf{x}}_{k|k} &= \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k(\mathbf{z}_k - \mathbf{H}\hat{\mathbf{x}}_{k|k-1}) && \text{(state update)}
\end{aligned}$$

KF Considerations

$$\begin{aligned}
\underbrace{\hat{\mathbf{x}}_{k|k-1}}_{n\times 1} &= \underbrace{\mathbf{F}_{k-1}}_{n\times n}\hat{\mathbf{x}}_{k-1|k-1} + \underbrace{\mathbf{G}_{k-1}}_{n\times j}\underbrace{\mathbf{u}_{k-1}}_{j\times 1}\\
\underbrace{\mathbf{P}_{k|k-1}}_{n\times n} &= \underbrace{\mathbf{Q}_{k-1}}_{n\times n} + \mathbf{F}_{k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}^{\mathsf{T}}{}_{k-1}\\
\underbrace{\mathbf{K}_k}_{n\times m} &= \mathbf{P}_{k|k-1}\underbrace{\mathbf{H}^{\mathsf{T}}}_{n\times m}\underbrace{[\mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^{\mathsf{T}} + \mathbf{R}_k]^{-1}}_{m\times m}\\
\mathbf{P}_{k|k} &= [\mathbf{I} - \mathbf{K}_k\mathbf{H}]\mathbf{P}_{k|k-1}\\
\hat{\mathbf{x}}_{k|k} &= \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k(\underbrace{\mathbf{z}_k}_{m\times 1} - \underbrace{\mathbf{H}}_{m\times n}\hat{\mathbf{x}}_{k|k-1} - \mathbf{H}\mathbf{G}_k\mathbf{u}_{k-1})
\end{aligned}$$

Ex: Kinematic KF: Tracking

- Consider a System with Constant Acceleration

$$\ddot{y} = -g$$
$$\dot{y} = gt + p_1$$
$$y = p_0 + p_1 t + \frac{gt^2}{2}$$

$$\begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ g \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{F_k} = \begin{bmatrix} 0 & t_s & \frac{t_s{}^2}{2} \\ 0 & 0 & t_s \\ 0 & 0 & 0 \end{bmatrix}$$

$$\hat{\mathbf{x}}_k = \mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1} + \mathbf{K}_k(\mathbf{z}_k - \mathbf{H}\mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1})$$

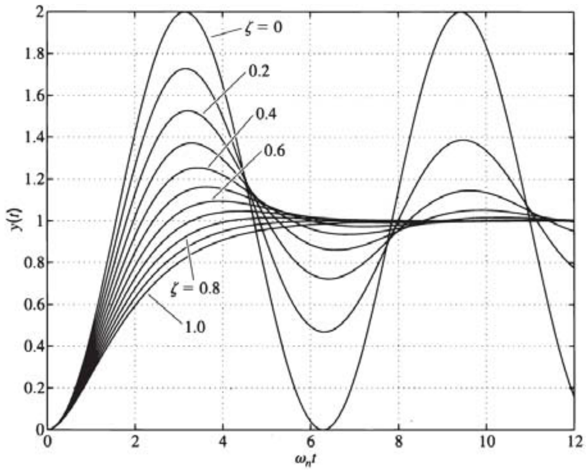
In Summary

- KF:
  - The true state (x) is separate from the measured (z)
  - Lets you **combine** prior controls knowledge with measurements to filter signals and find the truth
  - It **regulates** the covariance (P)
    - As P is the scatter between z and x
    - So, if P → 0, then z → x (measurements → truth)
- EKF:
  - Takes a Taylor series approximation to get a local "F" (and "G" and "H")

Example III: 2nd Order System Response

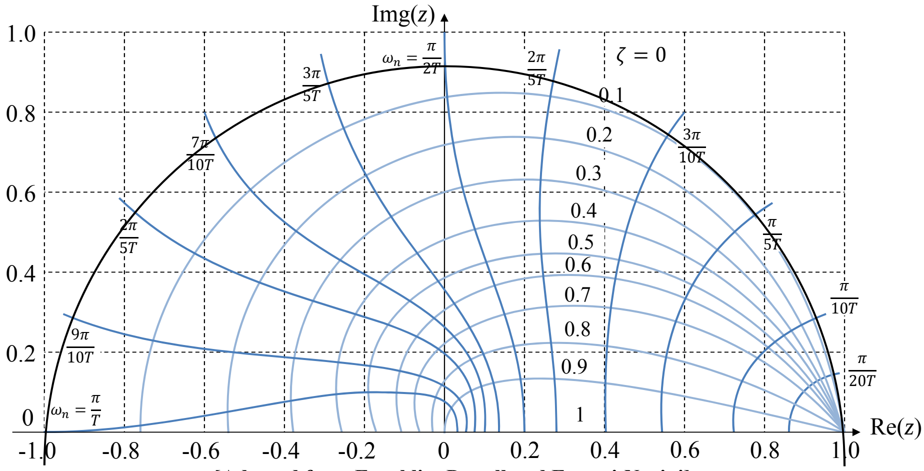
2nd Order System Response

- Response of a 2nd order system to increasing levels of damping:



Damping and natural frequency

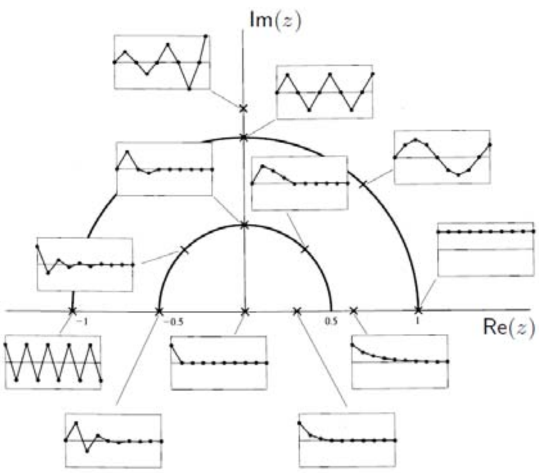
$$z = e^{sT} \text{ where } s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$



[Adapted from Franklin, Powell and Emami-Naeini]

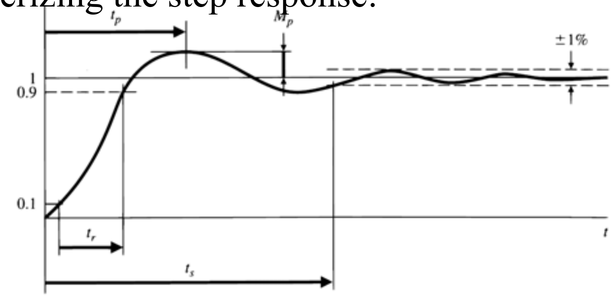
Pole positions in the z-plane

- Poles inside the unit circle are **stable**
- Poles outside the unit circle **unstable**
- Poles on the unit circle are oscillatory
- Real poles at 0 < z < 1 give exponential response
- Higher frequency of oscillation for larger
- Lower apparent damping for larer and r



2^{nd} Order System Specifications

Characterizing the step response:

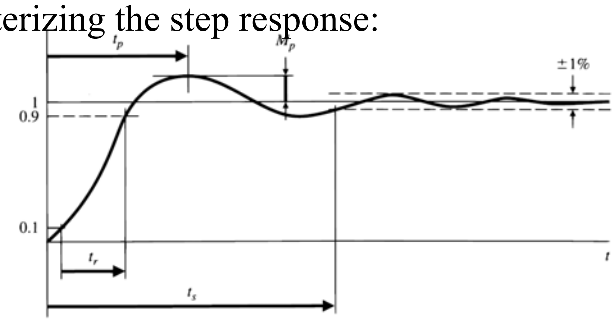


- Rise time (10% → 90%): $t_r \approx \frac{1.8}{\omega_0}$
- Overshoot: $M_p \approx \frac{e^{-\pi\zeta}}{\sqrt{1-\zeta^2}}$
- Settling time (to 1%): $t_s = \frac{4.6}{\zeta\omega_0}$
- Steady state error to unit step: e_{ss}
- Phase margin:

$$\phi_{PM} \approx 100\zeta$$

2^{nd} Order System Specifications

Characterizing the step response:



- Rise time (10% → 90%) & Overshoot:
  t_r, M_p → ζ, ω_0 : Locations of dominant poles
- Settling time (to 1%):
  t_s → radius of poles: $|z| < 0.01^{\frac{T}{t_s}}$
- Steady state error to unit step:
  e_{ss} → final value theorem $e_{ss} = \lim_{z \to 1} \{(z-1)F(z)\}$

→ SPNS - 9/27/2013 - Mech 4202 Lecture 10 ←

① Mass w/ force

$f \rightarrow$ [m]

$F = ma = m\ddot{x} \quad \therefore \quad \ddot{x} = \frac{f}{M}$

$v = \dot{x} \quad \therefore \quad \ddot{x} = \dot{v}$

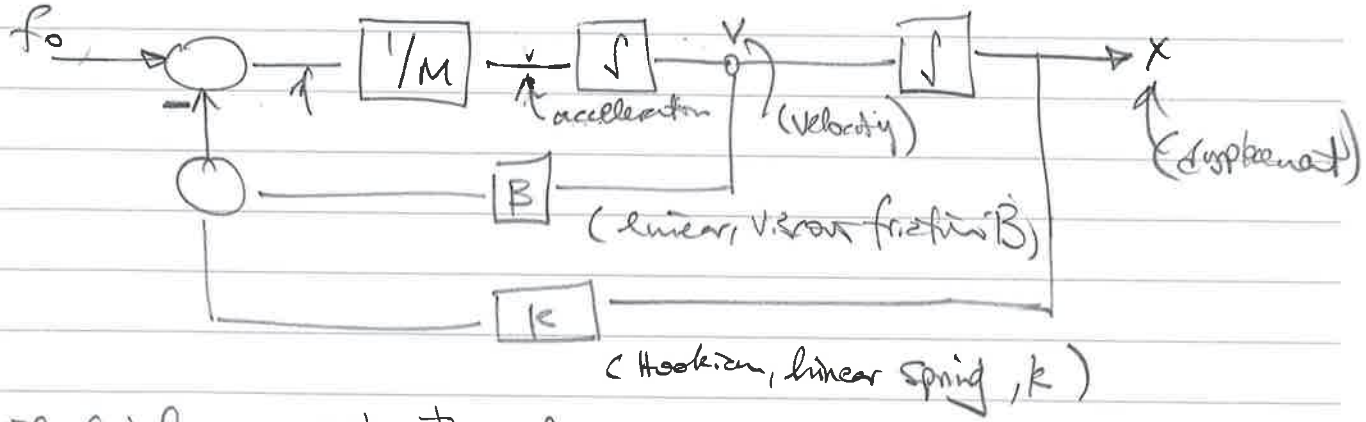
$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{f}{M}$

$\vec{X} = \begin{bmatrix} \dot{x} \\ x \end{bmatrix} \quad \dot{\vec{X}} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}$

② Electric Motor with inertial load

(torque) $\tau = k_1 i$ [Motor Torque]

(voltage) $V = k_2 \omega$ [Back emf]



Electrical power to the motor:

$p = vi = \frac{k_2 \omega \tau}{k_1}$

Mechanical power: $P_m = \omega \tau$

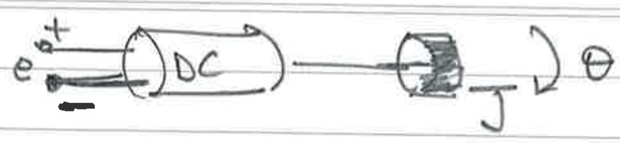
if $\eta = 100\% \rightarrow P_e = \frac{k_2}{k_1} P_m$

Note:

in practice:

$\therefore \frac{k_2}{k_1} > 1$

$\therefore \eta < 1$



e : Driving voltage (from battery)
v: Back emf

$$e - v = R\,i \qquad \text{(ohm's law)}$$

$$\tau = J\frac{d\omega}{dt} \qquad \text{[mechanical loading]}$$

$$J\frac{d\omega}{dt} = k_1\, i = \frac{k_1}{R}(e-v)$$

$$= \frac{k_1}{R}e - \frac{k_1 k_2}{R}\omega$$

$$\therefore \frac{d\omega}{dt} = -\frac{k_1 k_2}{JR}\omega + \frac{k_1}{JR}e$$

$$\omega = \frac{d\theta}{dt} \qquad \text{(! in 2D!!!!)}$$

in 3D: $\vec{V} = \vec{\omega} \times \vec{r}$

$$\vec{X} = \begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}\theta \\ \omega\end{bmatrix}$$

$$\begin{bmatrix}\dot{\theta} \\ \dot{\omega}\end{bmatrix} = \begin{bmatrix}\omega \\ \alpha\end{bmatrix} = \begin{bmatrix}0 & 1 \\ 0 & -\frac{k_1 k_2}{JR}\end{bmatrix}\begin{bmatrix}\theta \\ \omega\end{bmatrix} + \ldots$$

$$\begin{bmatrix}0 \\ \frac{k_1}{JR}\end{bmatrix}\begin{bmatrix}e\end{bmatrix}$$

Generalized Step

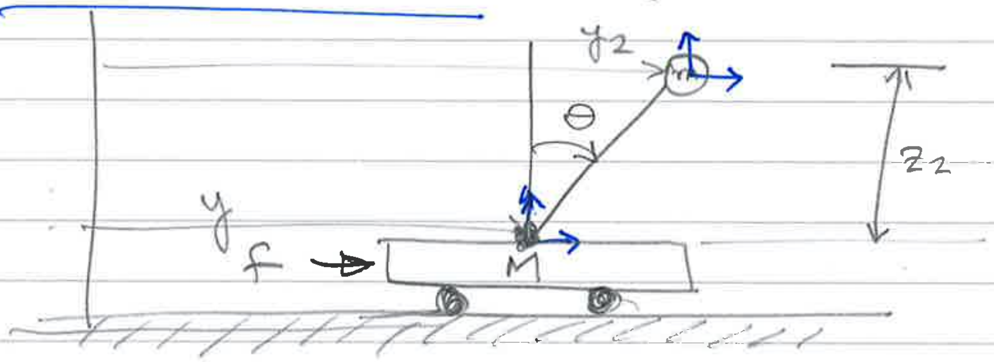
$$\delta(t) = \begin{cases} 1 & \text{for } t > t \\ 0 & \text{for } t \le t \end{cases}$$

LAGRANGE'S EQUATIONS

$L = T - V = KE - PE$ (in generalized coordinates)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q_i} = Q \qquad i = 1, 2, 3, \ldots, r$$

INVERTED PENDULUM (Moving on a cart)



$$T_1 = \frac{1}{2} M\dot{y}^2$$

$$T_2 = \frac{1}{2} m(\dot{y}_2^2 + \dot{z}_2^2)$$

$$y_2 = y + l\sin\theta = y + l s_1$$
$$z_2 = l\cos\theta = l c_1$$

$$\dot{y}_2 = \dot{y} + l\dot{\theta}c_1$$
$$\dot{z}_2 = -l\dot{\theta}s_1$$

$$T = T_1 + T_2 = \tfrac{1}{2} M \dot{y}^2 + \tfrac{1}{2} m \left[(\dot{y} + l\dot{\theta} c_1)^2 + l^2 \dot{\theta}^2 s_1^2 \right]$$
$$= \tfrac{1}{2} M \dot{y}^2 + \tfrac{1}{2} m \left[\dot{y}^2 + 2\dot{y}\dot{\theta} l c_1 + l^2 \dot{\theta}^2 \right]$$

Potential Energy

$$V = m g z_2 = m g l c_1$$

Lagrangian

$$L = T - V = \tfrac{1}{2}(M+m)\dot{y}^2 + m l c_1 \dot{y}\dot{\theta} + \tfrac{1}{2} m l^2 \dot{\theta}^2 - m g l c_1$$

I $\quad \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{y}}\right) - \dfrac{\partial L}{\partial y} = f$

II $\quad \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\theta}}\right) - \dfrac{\partial L}{\partial \theta} = 0$

$$\frac{\partial L}{\partial \dot{y}} = (M+m)\dot{y} + m l c_1 \dot{\theta} \qquad (1)$$

$$\frac{\partial L}{\partial y} = 0 \qquad (2)$$

$$\frac{\partial L}{\partial \dot{\theta}} = m l c_1 \dot{y} + m l^2 \dot{\theta} \qquad (3)$$

$$\frac{\partial L}{\partial \theta} = m g l s_1 - m l s_1 \dot{y}\dot{\theta} \qquad (4)$$