Table 2D.1 Numerical values of parameters in tank turret control

	Numerical value			
Parameter	Azimuth	Elevation		
K _v	94.3	94,3		
L_v	1.00	1.07		
J (ft-lb \cdot s ²)	7900.	2070.		
K _m ·	8.46×10^{6}	$1,96 \times 10^{6}$		
ω_m (rad/s)	45.9	17.3		
$K_{\Delta p}$	6.33×10^{-6}	3.86×10^{-5}		

Numerical data for a specific tank were found by Loh, Cheok, and Beck to be as given in Table 2D.1

A block-diagram representation of the dynamics represented by (2D.1) is shown in Fig. 2.9.

2.4 LAGRANGE'S EQUATIONS

The equations governing the motion of a complicated mechanical system, such as a robot manipulator, can be expressed very efficiently through the use of a method developed by the eighteenth-century French mathematician Lagrange. The differential equations that result from use of this method are known as *Lagrange's equations* and are derived from Newton's laws of motion in most textbooks on advanced dynamics.[2, 3]

Lagrange's equations are particularly advantageous in that they automatically incorporate the constraints that exist by virtue of the different parts of a system being connected to each other, and thereby eliminate the need for substituting one set of equations into another to eliminate forces and torques of constraint. Since they deal with scalar quantities (potential and kinetic energy) rather than with vectors (forces and torques) they also minimize the need for complicated vector diagrams that are usually required to define and resolve the vector quantities in the proper coordinate system. The advantages of Lagrange's equations may also turn out to be disadvantages, because it is necessary to identify the generalized coordinates correctly at the very beginning of the analysis of a specific system. An error made at this point may result in a set of differential equations that look correct but do not constitute the correct model of the physical system under investigation.

The fundamental principle of Lagrange's equations is the representation of the system by a set of generalized coordinates q_i (i = 1, 2, ..., r), one for each independent degree of freedom of the system, which completely incorporate the constraints unique to that system, i.e., the interconnections between the parts of the system. After having defined the generalized coordinates, the kinetic energy T is expressed in terms of these coordinates and their derivatives, and the



potential energy V is expressed in terms of the generalized coordinates. (The potential energy is a function of only the generalized coordinates and *not* their derivatives.) Next, the *lagrangian* function

$$L = T(q_1,\ldots,q_r,\dot{q}_1,\ldots,\dot{q}_r) - V(q_1,\ldots,q_r)$$

is formed. And finally the desired equations of motion are derived using Lagrange's equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = Q_i \qquad i = 1, 2, \dots, r$$
(2.11)

where Q_i denotes generalized forces (i.e., forces and torques) that are external to the system or not derivable from a scalar potential function.

Each of the differential equations in the set (2.11) will be a second-order differential equation, so a dynamic system with r degrees of freedom will be represented by r second-order differential equations. If one *state variable* is assigned to each generalized coordinate and another to the corresponding derivative, we end up with 2r equations. Thus a system with r degrees of freedom is of order 2r.

Example 2E Inverted pendulum on moving cart A typical application of Lagrange's equations is to define the motion of a collection of bodies that are connected together in some manner such as the inverted pendulum on a cart illustrated in Fig. 2.10.

It is observed that the motion of the system is uniquely defined by the displacement of the cart from some reference point, and the angle that the pendulum rod makes with respect to the vertical. Instead of using θ , we could use the horizontal displacement, say y_1 , of the bob relative to the pivot point, or the vertical height z_2 of the bob. But, whatever variables are used, it is essential to know that the system has only two degrees of freedom, and that the dynamics must be expressed in terms of the corresponding generalized coordinates.

The kinetic energy of the system is the sum of the kinetic energy of each mass. The cart is confined to move in the horizontal direction so its kinetic energy is





Figure 2.10 Inverted pendulum on moving cart.

STATE-SPACE REPRESENTATION OF DYN/

The bob can move in the horizontal and in the vertical direction so

$$T_2 = \frac{1}{2}m(\dot{y}_2^2 + \dot{z}_2^2)$$

But the rigid rod constrains z_2 and y_2

$$y_2 = y + l \sin \theta \qquad \dot{y}_2 = \dot{y} + l\dot{\theta} \cos \theta$$
$$z_2 = l \cos \theta \qquad \dot{z}_2 = -l\dot{\theta} \sin \theta$$

Thus

$$T = T_1 + T_2 = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}m[(\dot{y} + l\dot{\theta}\cos\theta)^2 + l^2\dot{\theta}^2\sin^2\theta]$$
$$= \frac{1}{2}M\dot{y}^2 + \frac{1}{2}m[\dot{y}^2 + 2\dot{y}\dot{\theta}l\cos\theta + l^2\dot{\theta}^2]$$

The only potential energy is stored in the bob

 $V = mgz_2 = mgl\cos\theta$

Thus the lagrangian is

$L = T - V = \frac{1}{2}(M + m)\dot{y}^{2} + ml\cos\theta\dot{y}\dot{\theta} + \frac{1}{2}ml^{2}\dot{\theta}^{2} - mgl\cos\theta$ (2E.1)

The generalized coordinates having been selected as (y, θ) , Lagrange's equations for this system are

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) - \frac{\partial L}{\partial y} = f$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0$$
(2E.2)

Now

$$\frac{\partial L}{\partial \dot{y}} = (M + m)\dot{y} + ml\cos\theta\dot{\theta}$$
$$\frac{\partial L}{\partial y} = 0$$
$$\frac{\partial L}{\partial \dot{\theta}} = ml\cos\theta\dot{y} + ml^2\dot{\theta}$$
$$\frac{\partial L}{\partial \theta} = mgl\sin\theta - ml\sin\theta\dot{y}\dot{\theta}$$

Thus (2E.2) become

$$(M+m)\ddot{y} + ml\cos\theta\theta - ml\theta^{2}\sin\theta = f$$

$$ml\cos\theta\ddot{y} + ml^{2}\ddot{\theta} - mgl\sin\theta = 0$$
(2E.3)

These are the exact equations of motion of the inverted pendulum on a cart shown in Fig. 2.10. They are nonlinear owing to the presence of the trigonometric terms $\sin \theta$ and $\cos \theta$ and the quadratic terms $\dot{\theta}^2$ and $\dot{y}\dot{\theta}$. If the pendulum is stabilized, however, then θ will be kept small. This justifies the approximations

$$\cos \theta \simeq 1$$
 $\sin \theta \simeq \theta$

We may also assume that $\dot{\theta}$ and \dot{y} will be kept small, so the quadratic terms are negligible. Using these approximations we obtain the linearized dynamic model

$$(M+m)\ddot{y} + ml\ddot{\theta} = f$$

$$m\ddot{y} + ml\ddot{\theta} - mg\theta = 0$$

(2E.4)



A state-variable representation corresponding to (2E.4) is obtained by defining the state vector $x = [y, \theta, \dot{y}, \dot{\theta}]^{\prime}$

 $\frac{dy}{dt} = \dot{y}$ $\frac{d\theta}{dt} = \dot{\theta}$ (2E.5)

constitute the first two dynamic equations and on solving (2E.4) for \ddot{y} and $\ddot{\theta}$, we obtain two more equations

$$\frac{d}{lt}(\dot{y}) = \ddot{y} = \frac{f}{M} - \frac{mg}{M}\theta$$

$$\frac{d}{dt}(\dot{\theta}) = \ddot{\theta} = -\frac{f}{Ml} + \left(\frac{M+m}{Ml}\right)g\theta$$
(2E.6)

The four equations can be put into the standard matrix form

$$\dot{x} = Ax + Bu$$

with

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -mg/M & 0 & 0 \\ 0 & (M+m)a/Ml & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1/M \\ -1/Ml \end{bmatrix}$$

and

u = f = external force

A block-diagram representation of the dynamics (2E.5) and (2E.6) is shown in Fig. 2.11.



Figure 2.11 Block diagram of dynamics of inverted pendulum on moving cart.

2.5 RIGID BODY DYNAMICS

The motion of a single rigid body has six dynamic degrees of freedom: three of these define the location of a reference point (usually the center of mass) in the body, and three define the orientation (attitude) of the body. Since each of the six degrees of freedom takes two state variables (one position and one velocity) a total of 12 first-order differential equations are required to completely describe the motion of the body. In most applications, however, not all of these 12 state variables are of interest and not all the differential equations are needed. In a gyroscope, for example, only the orientation is of interest.

The motion of a rigid body is, of course, governed by the familiar newtonian laws of motion

$$\frac{d\vec{p}}{dt} = \vec{f} \tag{2.12}$$

$$\frac{d\vec{h}}{dt} = \vec{\tau} \tag{2.13}$$

where $\vec{p} = [p_x, p_y, p_z]'$ is the linear momentum of the body

 $\vec{h} = [h_x, h_y, h_z]'$ is the angular momentum of the body

 $\vec{f} = [f_x, f_y, f_z]'$ is force acting on the body

 $\vec{\tau} = [\tau_x, \tau_y, \tau_z]'$ is torque acting on the body

It is important to understand that (2.12) and (2.13) are valid only when the axes along which the motion is resolved are an inertial frame of reference, i.e., they are neither accelerating nor rotating. If the axes are accelerating linearly or rotating, then (2.12) and (2.13) must be modified to account for the motion of the reference axes.

The rotational dynamics of a rigid body are more complicated than the translational dynamics for several reasons: the mass M of a rigid body is a scalar, but the moment of inertia J is a 3×3 matrix. If the body axes are chosen to coincide with the "principal axes," the moment of inertia matrix is diagonal; otherwise the matrix J has off-diagonal terms. This is not the only complication, however, or even the main one. The main complication is in the description of the attitude or orientation of the body in space. To define the orientation of the body in space, we can define three axes (x_B, y_B, z_B) fixed in the body, as shown in Fig. 2.12. One way of defining the attitude of the body is to define the angles between the body axes and the inertial reference axes (x_I, y_I, z_I) . These angles are not shown in the diagram. Not only are they difficult to depict in a two-dimensional picture, but they are not always defined the same way. In texts on classical mechanics, the orientation of the body is defined by a set of three angles, called *Euler angles*, which describe the orientation of a set of nonorthogonal axes fixed in the body with respect to the inertial reference axes. In aircraft and space mechanics it is now customary to define the orientation of a set of orthogonal axes in the body (body axes) with respect to the inertial reference.





Suppose the body axes are initially aligned with the inertial reference axes. Then, the following sequence of rotations are made to bring the body axes into general position:

First, a rotation ψ (yaw) about the z axis Second, a rotation θ (pitch) about the resulting y axis Third, a rotation ϕ (roll) about the resulting x axis

By inspection of the diagrams of Fig. 2.13 we see that

$$\begin{bmatrix} x_{B1} \\ y_{B1} \\ z_{B1} \end{bmatrix} = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_I \\ y_I \\ z_I \end{bmatrix}$$
(2.14)
$$\begin{bmatrix} x_{B2} \\ y_{B2} \\ z_{B2} \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} x_{B1} \\ y_{B1} \\ z_{B1} \end{bmatrix}$$
(2.15)
$$\begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix} = \begin{bmatrix} x_{B3} \\ y_{B3} \\ z_{B3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} x_{B2} \\ y_{B2} \\ z_{B2} \end{bmatrix}$$
(2.16)

Thus we see that

$$\begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix} = T_{BI} \begin{bmatrix} x_I \\ y_I \\ z_I \end{bmatrix}$$

1





Figure 2.13 Sequence of rotations of body axes from reference to "general" orientation (z axis down in aircraft convention). (a) Axes in reference position; (b) First rotation--about z axis--yaw (ψ); (c) Second rotation---about y axis---pitch (θ); (d) Third and final rotation----about x axis---roll (ϕ).

where T_{BI} is the matrix that rotates the body axes from reference position, and is the product of the three matrices in (2.14)-(2.16).

	٦l	0	0]	$\int \cos \theta$	0	$-\sin\theta$	$\int \cos \psi$	$\sin\psi$	0	
$T_{BI} =$	0	$\cos \phi$	$\sin \phi$	0	1	0	$-\sin\psi$	$\cos\psi$	0	(2.17)
	0	$-\sin\phi$	$\cos \phi$	$\sin \theta$	0	$\cos \theta$	0	0	1	

Each factor of T_{BI} is an orthogonal matrix and hence T_{BI} is orthogonal, i.e.,

$$T_{IB} = T_{BI}^{-1} = T'_{BI} (2.18)$$

Note that $T_{BI}^{-1} = T_{IB}$ is the matrix that returns the body axes from the general position to the reference position.

Note that the order of rotations implicit in T_{BI} is important: the three matrices in (2.17) do not commute.

Since any vector in space can be resolved into its components in body axes or in inertial axes, we can use the transformation (2.17) to obtain the components of a vector in one set of axes, given its components in the other. In particular suppose \vec{a} is any vector in space. When it is resolved into components along an inertial reference we attach the subscript I; when it is resolved in body axes, we attach the subscript B

$$\vec{a}_{I} = \begin{bmatrix} a_{xI} \\ a_{yI} \\ a_{zI} \end{bmatrix} \qquad \vec{a}_{B} = \begin{bmatrix} a_{xB} \\ a_{yB} \\ a_{zB} \end{bmatrix}$$





Using (2.17) we obtain

$$\vec{a}_B = T_{BI}\vec{a}_I \tag{2.19}$$

This relationship can be applied to (2.13) for the angular motion of a rigid bod and, as we shall see later, for describing the motion of an aircraft along rotatin, body axes.

In the case of a rigid body, the angular momentum vector is

$$\dot{h} = J\vec{\omega} \tag{2.20}$$

where J is the moment of inertia matrix and $\vec{\omega}$ is the angular velocity vector. I the axes along which \vec{h} is resolved are defined to be coincident with the *physica* principal axes of the body, then J is a diagonal matrix. Thus when \vec{h} is resolved along principal body axes, we get from (2.17)

$$\vec{h}_B = \begin{bmatrix} J_x \omega_x \\ J_y \omega_y \\ J_z \omega_z \end{bmatrix}$$
(2.21)

But (2.13) holds only when the vector \vec{h} is measured with respect to an inertia reference: In the notation established above

$$\frac{d\vec{h}_I}{dt} = \frac{d}{dt} \left(T_{IB} \vec{h}_B \right) = \vec{\tau}_I \tag{2.22}$$

The transformation T_{IB} , however, is not constant. Hence (2.22) must be written

$$T_{IB}\vec{h}_B + \dot{T}_{IB}\vec{h}_B = \vec{\tau}_I$$

or, multiplying both sides by $T_{BI} = T_{IB}^{-1}$:

$${}_{B} + T_{BI} \dot{T}_{IB} \vec{h}_{B} = T_{BI} \vec{\tau}_{I} = \vec{\tau}_{B}$$
(2.23)

which, in component form can be written

h

$$\begin{bmatrix} J_{x}\dot{\omega}_{xB} \\ J_{y}\dot{\omega}_{yB} \\ J_{z}\dot{\omega}_{zB} \end{bmatrix} + T_{BI}\dot{T}_{IB} \begin{bmatrix} J_{x}\omega_{xB} \\ J_{y}\omega_{yB} \\ J_{z}\omega_{zB} \end{bmatrix} = \begin{bmatrix} \tau_{xB} \\ \tau_{yB} \\ \tau_{zB} \end{bmatrix}$$
(2.24)

These differential equations relate the components of the angular velocity vector, $\vec{\omega}$ projected onto rotating body axes

 $\vec{\omega}_B = [\omega_{xB}, \, \omega_{yB}, \, \omega_{zB}]'$

to the torque vector also projected along body axes. To complete (2.24) we need the matrix $T_{BI}T_{IB}$. It can be shown that

$$\dot{T}_{IB} = T_{IB} \begin{bmatrix} 0 & -\omega_{zB} & \omega_{yB} \\ \omega_{zB} & 0 & -\omega_{xB} \\ -\omega_{yB} & \omega_{xB} & 0 \end{bmatrix}$$
(2.25)

So that

$$T_{BI}\check{T}_{IB} = \begin{bmatrix} 0 & -\omega_{zB} & \omega_{yB} \\ \omega_{zB} & 0 & -\omega_{xB} \\ -\omega_{yB} & \omega_{xB} & 0 \end{bmatrix}$$
(2.26)

(See Note 2.1.)

Hence (2.24) becomes

$$J_{x}\dot{\omega}_{xB} + (J_{z} - J_{y})\omega_{yB}\omega_{zB} = \tau_{xB}$$

$$J_{y}\dot{\omega}_{yB} + (J_{x} - J_{z})\omega_{xB}\omega_{zB} = \tau_{yB}$$

$$J_{z}\dot{\omega}_{zB} + (J_{y} - J_{x})\omega_{xB}\omega_{yB} = \tau_{zB}$$
(2.27)

These are the famous *Euler equations* that describe how the body-axis components of the angular velocity vector evolve in time, in response to torque components in body axes.

In order to completely define the attitude (orientation), we need to relate the rotation angles ϕ , θ , and ψ to the angular velocity components. One way—not the easiest, however—of obtaining the required relations is via (2.17) and (2.25). It can be shown that

$$\begin{split} \dot{\phi} &= \omega_x + (\omega_y \sin \phi + \omega_z \cos \phi) \tan \theta \\ \dot{\theta} &= \omega_y \cos \phi - \omega_z \sin \phi \end{split} \tag{2.28}$$
$$\dot{\psi} &= (\omega_x \sin \phi + \omega_y \cos \phi) / \cos \theta \end{split}$$

These relations, also nonlinear, complete the description of the rigid body dynamics.

Example 2F The gyroscope One of the most interesting applications of Euler's equations is to the study of the gyroscope. This device (also the spinning top) has fascinated mathematicians and physicists for over a century. (See Note 2.2.) And the gyroscope is an extremely useful sensor of aircraft and spacecraft motion. Its design and control has been an important technological problem for half a century.

In an ideal gyroscope the rotor, or "wheel," is kept spinning at a constant angular velocity. (A motor is provided to overcome the inevitable friction torques present even in the best of instruments. The precise control of wheel speed is another important control problem.) Suppose that the axis through the wheel is the body z axis. We assume that τ_{zB} is such that $\dot{\omega}_{zB} = 0$, i.e., that

$$H_z = J_z \omega_z = \text{const} \tag{2F.1}$$

 $(J_z$ is called the "polar" moment of inertia in gyro parlance.) We can also assume that the gyroscope wheel is a "true" wheel: that the z axis is an axis of symmetry, and hence that

 $J_x = J_y = J_d$ (the "diametrical" moment of inertia)

The first two equations of (2.27) then become

$$\begin{split} \dot{\omega}_{xB} &+ \frac{H}{J_d} \, \omega_{yB} = \frac{\tau_x}{J_d} \\ \dot{\omega}_{yB} &- \frac{H}{J_d} \, \omega_{xB} = \frac{\tau_y}{J_d} \end{split}$$

(2F.2)

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Figure 2.14 Two-degrees-of-freedom gyro wheel.

where

$$H = H_z \left(1 - \frac{J_d}{J_z} \right)$$

To use a gyro as a sensor, the wheel is mounted in an appropriate system of gimbals which permit it to move with respect to the outer case of the gyro. In a two-axis gyro, the wheel is permitted two degrees of freedom with respect to the case, as depicted in Fig. 2.14. The case of the gyro is rigidly attached to the body whose motion is to be measured.

The range of motion of the wheel about its x and y body axes relative to the gyro case is very small (usually a fraction of a degree). Hence the gyro must be "torqued" about the axes in the plane normal to the spin axis to make the wheel keep up with its case, and as we shall see shortly, the torque required to do this is a measure of the angular velocity of the case.

Since the motion of the wheel relative to the case is very small, we do not need equations like (2.27) to relate the angular displacements of the gyro wheel from its null positions in the case. We can write

$$\begin{split} \dot{\delta}_{x} &= \omega_{xB} - \omega_{xE} \\ \dot{\delta}_{y} &= \omega_{yB} - \omega_{yE} \end{split} \tag{2F.3}$$

where ω_{xE} and ω_{yE} are the external angular velocities that the gyro is to measure.

These equations, together with (2F.2), constitute the basic equations of an ideal gyro. A block-diagram representation of (2F.2) and (2F.3), and a closed-loop feedback system for controlling the gyro is shown in Fig. 2.15. The feedback system shows the control torques generated as functions of the displacements δ_x and δ_y . These displacements can be measured by means of "pick-offs"—small magnetic sensors located on the case and capable of measuring small tilts of the wheel. The control torque needed to drive the "pick-off angles" δ_x and δ_y to zero can also be generated magnetically. In some designs the pick-off and torquer functions can be combined in a single device. The control system is designed to drive the angular displacements δ_x and δ_y to zero. If this is accomplished

$$\omega_{xB} = \omega_{xE} \qquad \omega_{yB} = \omega_{yE} \tag{2F.4}$$



Figure 2.15 Block diagram of two-axis gyro dynamics showing "capture" control system.

If the angular velocity components ω_{xB} and ω_{yB} are constant

$$\tau_{x} = H\omega_{yB} = H\omega_{yE}$$

$$\tau_{y} = -H\omega_{xB} = -H\omega_{xE}$$
(2F.5)

where H is a constant of the gyro. If this constant is accurately calibrated, and if the input torque to the gyro is accurately metered, then the steady state torques about the respective axes that keep the wheel from tilting relative to its case (i.e., "capture" the wheel) are proportional to the measured external angular velocity components.

The control system that keeps the wheel captured is an important part of every practical gyro. Some of the issues in the design of such a control system will be the subject of problems in later chapters.

The differential equations of (2F,2) are idealized to the point of being all but unrealistic. In addition to the control torques acting on the gyro, other torques, generated internal to the gyro, are also inevitably present. These include damping torques (possibly aerodynamic). And in a so-called tuned-rotor gyro, the gimbals are implemented by a special flexure hinge which produces small but not insignificant spring torques. When these torques are included, (2F.2) becomes

$$\begin{split} \dot{\omega}_{xB} &= \frac{H}{J_d} \,\omega_{yB} - \frac{B}{J_d} \left(\omega_{xB} - \omega_{xE} \right) - \frac{K_D}{J_d} \,\delta_x - \frac{K_Q}{J_d} \,\delta_y + \frac{\tau_x}{J_d} \\ \dot{\omega}_{yB} &= -\frac{H}{J_d} \,\omega_{xB} - \frac{B}{J_d} \left(\omega_{yB} - \omega_{yE} \right) - \frac{K_D}{J_d} \,\delta_y + \frac{K_Q}{J_d} \,\delta_x + \frac{\tau_y}{J_d} \end{split}$$



Note that the damping coefficients D in both axes are assumed equal and that the "spring" matrix

$$K = \begin{bmatrix} -K_D & -K_Q \\ K_Q & -K_D \end{bmatrix}$$

has a special kind of symmetry. This form of the matrix is justified by the physical characteristics of typical tuned-rotor gyros.

2.6 AERODYNAMICS

One of the most important applications of state-space methods is in the design of control systems for aircraft and missiles.

The forces (except for gravitation) and moments on such vehicles are produced by the motion of the vehicle through the air and are obtained, in principle, by integrating the aerodynamic pressure over the entire surface of the aircraft. Computer programs for actually performing this integration numerically are currently available. In an earlier era this was accomplished by approximate analysis done by skillful aerodynamicists, and verified by extensive wind-tunnel testing. (Wind-tunnel tests are performed to this day, notwithstanding the computer codes.)

Several textbooks, e.g., [4, 5], are available which give an exposition of the relevant aerodynamic facts of interest to the control system designer. The aerodynamic forces and moments are complicated, nonlinear functions of many variables and it is barely possible to scratch the surface of this subject here. The purpose of this section is to provide only enough of the principles as are needed to motivate the design examples to be found later on in the book.

The aerodynamic forces and moments depend on the velocity of the aircraft relative to the air mass. In still air (no winds) they depend on the velocity of the aircraft along its own body axes: the orientation of the aircraft is not relevant in determining the aerodynamic forces and moments. But, since the natural axes for resolving the aerodynamic forces and moments are moving (rotating and accelerating), it is necessary to formulate the equations of motion in the moving coordinate system.

The rotation motion of a general rigid body has been given in (2.24). In aircraft terminology the projections of the angular velocity vector on the body x, y, and z axes have standard symbols:

$$\omega_x = p$$
 (roll rate)
 $\omega_y = q$ (pitch rate) (2.29)
 $\omega_z = r$ (yaw rate)

(The logic of using three consecutive letters of the alphabet (p, q, r) to denote the projections of the angular velocity vector on the three consecutive body axes is unassailable. But the result is "amnemonic" (hard to remember): p does not represent pitch rate and r does not represent roll rate.)

Thus, assuming that the body axes are the principal axes of the aircraft, the rotational dynamics are expressed as

$$\dot{p} = \frac{L}{J_x} - \frac{J_z - J_y}{J_x} qr$$

$$\dot{q} = \frac{M}{J_y} - \frac{J_x - J_z}{J_y} pr$$

$$\dot{r} = \frac{N}{J_z} - \frac{J_y - J_x}{J_z} pq$$
(2.30)

where L, M, and N are the aerodynamic moments about the body x, y, and z axes respectively. Thus L is the rolling moment, M is the pitching moment, and N is the yawing moment. These are functions of various dynamic variables, as explained later.

To define the translational motion of an aircraft it is customary to project the velocity vector onto body fixed axes

$$\vec{v}_B = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$
(2.31)

where u, v, and w are the projections of the vehicle velocity vector onto the body x, y, and z axes. The linear momentum of the body, in an inertial frame, is

$$\vec{p} = m\vec{v}_I = mT_{IB}\vec{v}_B$$

Hence, the dynamic equations for translation are

$$\frac{d}{dt}(mT_{IB}\vec{v}_B) = m\left(T_{IB}\frac{d\vec{v}_B}{dt} + \dot{T}_{IB}\vec{v}_B\right) = \vec{f}_I \qquad (2.32)$$

where \vec{f}_I are the external forces acting on the aircraft referred to an inertial frame. Proceeding as we did in developing (2.24) we find that

$$\frac{d\vec{v}_B}{dt} = -T_{BI}\dot{T}_{IB}\vec{v}_B + \frac{1}{m}\vec{f}_B$$
(2.33)

where $\vec{f}_B = T_{BI}\vec{f}_I$ is the force acting on the aircraft resolved along the bodyfixed axes and

$$T_{BI}\dot{T}_{IB} = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}$$
(2.34)

as given by (2.26) but using the p, q, r notation defined in (2.29).



In component form (2.33) becomes

$$\dot{u} = rv - qw + \frac{1}{m} f_{xB}$$

$$\dot{v} = -ru + pw + \frac{1}{m} f_{yB}$$
(2.35)

$$\dot{w} = qu - pv + \frac{1}{m} f_{zB}$$

where f_{xB} , f_{yB} , and f_{zB} are the total forces (engine, aerodynamic, and gravitational) acting on the body. Since the aircraft axes are not in general in the direction of the gravity vector, each component f_{xB} , f_{yB} , and f_{zB} will have a term due to gravity. In addition to the force of gravity, there is the thrust force produced by the aircraft engine—generally assumed to act along the vehicle x axis—and the aerodynamic forces—the lift and drag forces. The acceleration terms rv, qw, etc., are Coriolis accelerations due to the rotation of the body axes.

Complete dynamic equations of the vehicle consist of (2.30) which give the angular accelerations, (2.35) which give the linear accelerations, (2.28) which give the angular orientation, and finally the equations for the vehicle position:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \dot{\vec{r}} = \vec{v}_I = T_{IB} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$
(2.36)

This system of 12 first-order differential equations, with the moments and forces evaluated as functions of whatever they depend upon constitute the complete six-degrees-of-freedom description of the aircraft behavior.

The aerodynamic forces and moments all depend on the dynamic pressure

$$Q = \frac{1}{2}\rho V^2 \tag{2.37}$$

where ρ is the air density and

$$V = (u^2 + v^2 + w^2)^{1/2}$$

is the speed of the aircraft. (Dynamic pressure has the dimension of force per unit area.) Thus the aerodynamic forces and moments can be expressed in the form

$$f_{xA} = QAC_x$$

$$f_{yA} = QAC_y$$

$$f_{zA} = QAC_z$$

$$L = lQAC_L$$

$$M = lQAC_M$$

$$N = lOAC_N$$
(2.38)

where C_x , C_y , C_z , C_L , C_M , C_N , are dimensionless aerodynamic "coefficients," A



is a reference area (usually the frontal area of the vehicle), and l is a reference length. (In some treatments different reference lengths are used for roll, pitch, and yaw.)

The aerodynamic coefficients in turn are functions of the vehicle velocity (linear and angular) components, and, for movable control surfaces, also functions of the deflections of the surfaces from their positions of reference. The variables of greatest influence on the coefficients are the vehicle speed (or, more precisely, the Mach number), the angle-of-attack α and the side-slip angle β . These, respectively, define the direction of the velocity vector relative to the vehicle body axes; α is the angle that the velocity vector makes with respect to the longitudinal axis in the pitch direction and β is the angle it makes with respect to the longitudinal axis in the yaw direction. (See Fig. 2.16.) From the figure

$$\alpha = \tan^{-1} \left(\frac{w}{(u^2 + v^2)^{1/2}} \right) \simeq \frac{w}{u}$$

$$\beta = \tan^{-1} \left(\frac{v}{u} \right) \simeq \frac{v}{u}$$
(2.39)

with the approximate expressions being valid for small angles.

For purposes of control system design, the aircraft dynamics are frequently linearized about some operating condition or "flight regime," in which it is assumed that the aircraft velocity and attitude are constant. The control surfaces and engine thrust are set, or "trimmed," to these conditions and the control system is designed to maintain them, i.e., to force any perturbations from these conditions to zero.

If the forward speed is approximately constant, then the angle of attack and angle of side slip can be used as state variables instead of w and v, respectively.



Figure 2.16 Definitions of angle-of-attack α and side-slip angle β .



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	Longitudinal	Lateral
Rates	α : angle of attack q: pitch rate Δu : change in speed	 β: side slip angle p: roll rate r: yaw rate
Positions	θ: pitch z: altitude	 φ: roll angle ψ: yaw angle x: forward displacement y: cross-track displacement
Controls	δ_E : elevator deflection	δ_A : aileron deflection δ_R : rudder deflection

Also in studying small perturbations from trim conditions it is customary to separate the longitudinal motion from the lateral motion. In many cases the lateral and longitudinal dynamics are only lightly coupled, and the control system can be designed for each channel without regard to the other. The variables are grouped as shown in Table 2.1.





The aircraft pitch motion is typically controlled by a control surface called the *elevator* (or by canards in the front of the vehicle). The roll is controlled by a pair of ailerons, and the yaw is controlled by a rudder. These are also shown in Table 2.1.

The function of most control system designs is to regulate *small* motions rather than to control absolute position (x, y, and z). Thus the inertial position is frequently not included in the state equations. This leaves nine equations, four in the longitudinal channel and five in the lateral channel. These can be written in the following form:

Longitudinal dynamics (See Fig. 2.17)

$$\Delta \dot{u} = X_{u} \Delta u + X_{\alpha} \alpha - g\theta + X_{E} \delta_{E}$$

$$\dot{\alpha} = \frac{Z_{u}}{V} \Delta u + \frac{Z_{\alpha}}{V} \alpha + q + \frac{Z_{E}}{V} \delta_{E}$$

$$\dot{q} = M_{u} \Delta u + M_{\alpha} \alpha + M_{q} q + M_{E} \delta_{E}$$

$$\dot{\theta} = q$$
(2.40)

Lateral dynamics (See Fig. 2.18)

$$\dot{\beta} = \frac{Y_{\beta}}{V}\beta + \frac{Y_{p}}{V}p + \left(\frac{Y_{r}}{V} - 1\right)r + \frac{g}{V}\phi + \frac{Y_{A}}{V}\delta_{A} + \frac{Y_{R}}{V}\delta_{R}$$

$$\dot{p} = L_{\beta}\beta + L_{p}p + L_{r}r + L_{A}\delta_{A} + L_{R}\delta_{R}$$

$$\dot{r} = N_{\beta}\beta + N_{p}p + N_{r}r + N_{A}\delta_{A} + N_{R}\delta_{R}$$

$$\dot{\phi} = p$$

$$\dot{\psi} = r$$
(2.41)

The symbols X, Y, Z, L, M, and N, with subscripts have become fairly standardized in the field of aircraft and missile control, although the sign conventions often differ from one user to another, which can often cause consternation. The symbols with the capital-letter subscripts, E, A, and R (for elevator, ailerons, and rudder), however, are not standard. It is customary to use cumbersome double subscript notation for these quantities.

2.7 CHEMICAL AND ENERGY PROCESSES

It is often necessary to control large industrial processes which involve heat exchangers, chemical reactors, evaporators, furnaces, boilers, driers, and the like.

Because of their large physical size, such processes have very slow dynamic behavior—measured on a scale of minutes or hours rather than seconds as in the case for aircraft and instrument controls. Such processes are often slow

