# CHAPTER NINE

# LINEAR, QUADRATIC OPTIMUM CONTROL

# 9.1 WHY OPTIMUM CONTROL?

In the previous chapters we learned how to design a compensator for a single-input, single-output process which places the closed-loop poles wherever we want them to be (assuming the process is controllable and observable). Since the closed-loop poles determine the speed (bandwidth) and damping of the response, isn't this enough? Why should we want to go any farther? There are several good reasons.

The first reason for seeking an optimum controller is that in a multipleinput or multiple-output system, the pole-placement technique described in the earlier chapters does not completely specify the controller or compensator parameters (gains). Consider, for example, a kth-order plant with m inputs and the entire state vector accessible for feedback. A nondynamic controller has km parameters to be determined, but only k possible closed-loop pole locations. Thus we have to set m times as many parameters as there are poles; there are infinitely many ways by which the same closed-loop poles can be attained. Which way is best? What algorithm can be used to determine the feedback gains? From a practical standpoint, of course, the availability of more adjustable parameters than the minimum number needed to achieve the desired closed-loop pole location is a great benefit because other things can be accomplished besides placing the closed-loop poles. But the absence of a definitive algorithm for determining a unique control law is a detriment to the system designer who does not know how to handle this "embarrassment of riches." By choosing a control law to optimize performance (in the precise sense to be defined shortly) this embarrassment is avoided.

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A more cogent reason for seeking an optimum controller is that the designer may not really know the desirable closed-loop pole locations. Choosing pole locations far from the origin may give very fast dynamic response but require control signals that are too large to be produced with the available power source. Use of gains that would be able to produce these signals, in the absence of power limitations, could cause the control signals to exceed physical limits (i.e., to "saturate"). In such cases the closed-loop dynamic behavior will not be as predicted by the linear analysis, and may even be unstable. To avoid these problems it often is necessary to limit the speed of response to that which can be achieved without saturation. Another reason for limiting speed of response is a desire to avoid problems of noise that typically accompany high-gain systems. The engineer who has acquired extensive experience with a particular type of process generally has an intuitive "feel" about the proper closed-loop pole locations. But, faced with an unfamiliar process to control and a lack of time to acquire the necessary insight, the engineer will appreciate a design method that can provide an initial design while insight is developed. The optimization theory, to be developed in this chapter, can serve this purpose.

Still another reason for using optimum control theory for design is that the process to be controlled may not be controllable, in the sense defined in Chap. 5. There may be some subspace of the process state-space in which the state vector cannot be moved around by application of suitable control signals. The dynamic behavior in that subspace is not subject to control and hence not all the poles of the closed-loop system can be placed at will. Hence design by pole placement will not work. But, by use of optimum control theory, and not demanding impossible behavior, it is possible to design a control system to control as much as can be controlled. If the behavior of the uncontrollable part is stable, the overall system will behave in an acceptable manner.

# 9.2 FORMULATION OF THE OPTIMUM CONTROL PROBLEM

The dynamic process considered here as elsewhere in this text is, as usual, characterized by the vector-matrix differential equation

$$\dot{\kappa} = Ax + Bu \tag{9.1}$$

where x is the process state, u is the control input, and A and B are known matrices. Again, as before, we seek a linear control law

$$u(t) = -Gx(t) \tag{9.2}$$

where G is a suitable gain matrix. Here, however, instead of seeking a gain matrix to achieve specified closed-loop pole locations, we now seek a gain to minimize a specified performance criterion V (or "cost function") expressed as the integral of a quadratic form in the state x plus a second quadratic form in

the control u; i.e.,

$$V = \int_{\tau}^{T} \left[ x'(\tau) Q(\tau) x(\tau) + u'(\tau) R u(\tau) \right] d\tau$$
(9.3)

where Q and R are symmetric matrices.

Some explanatory remarks about this performance criterion are in order before we attempt to find the optimum gain matrix G.

First, we note that minimization of V also minimizes  $\rho V$  where  $\rho$  is any positive constant. So the problem is not altered by multiplying V by any positive constant. Often the constant 1/2 is used in front of V to simplify expressions resulting in other developments. (See Note 9.1.)

Second, regarding the limits on the integral, the lower limit t is identified as the *present* time, and the upper limit T is the *terminal* time, or *final* time. The time difference T - t is the control interval, or "time-to-go." If the terminal time T is finite and fixed, the time-to-go keeps decreasing to zero, at which time the control process ends. This situation is characteristic of missile guidance problems, as will be discussed in an example below. The more customary case, however, is that in which the terminal time is infinite. In this case we are interested in the behavior of the process "from now on," including the steady state. This is precisely the case addressed by pole placement, and is the case that will receive the major portion of our attention subsequently.

Finally, consider the weighting matrices Q and R. These are often called the *state weighting* matrix and *control weighting* matrix, respectively. We are about to derive a "recipe" for finding the control gain matrix G in terms of these weighting matrices. In other words, we can plug the matrices Q and R—along with the matrices A and B that define the dynamic process—into a computer program and direct it to find G. If the process is controllable and Q and R are suitable, the computer will not fail to find G. (This is not to say that the calculation is a numerically trivial problem—far from it—but only that the problem of determining G once A, B, Q, and R are given, is not a control system design problem but a problem in numerical analysis.)

The question of concern to the control system designer is the selection of the weighting matrices Q and R. In candor one must admit that minimization of a quadratic integral of the form of (9.3) is rarely the true design objective. The problem, however, is that the true design objective often cannot be expressed in mathematical terms. And even in those instances when the design objective is amenable to mathematical expression, it is usually all but impossible to solve for the optimum control law. Expression of the design objective in the form of a quadratic integral is a practical compromise between formulating the real problem that cannot be solved, and formulating a somewhat artificial problem that can be solved easily. The need for such compromises arises in many contexts, and the control system designer should not feel guilty about being acquiescent to the need.

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In the performance or cost function defined by (9.3) two terms contribute to the integrated cost of control: the quadratic form x'Qx which represents a

penalty on the deviation of the state x from the origin and the term u'Ru which represents the "cost of control." This means, of course, that the desired state is the origin, not some other state. (In Chap. 5 we studied how it is possible to formulate a problem with a nonzero desired state in the form of a regulator problem. This discussion will be resumed in Sec. 9.6.) The weighting matrix Q specifies the importance of the various components of the state vector relative to each other. For example, suppose that  $x_1$  represents the system error, and that  $x_2, \ldots, x_k$  represent successive derivatives, i.e.,

$$\dot{x}_2 = \dot{x}$$
$$\dot{x}_3 = \ddot{x}$$
$$\dots$$
$$\dot{x}_k = x^{(k-1)}$$

If only the error and none of its derivatives are of concern, then we might select a state weighting matrix

$$Q = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$
(9.4)

which will yield the quadratic form

$$x'Qx = x_1^2$$

But the choice of (9.4) as a state weighting matrix may lead to a control system in which the velocity  $x_2 = \dot{x}$  is larger than desired. To limit the velocity, the performance integral might include a velocity penalty, i.e.,

$$x'Qx = x_1^2 + c^2 x_2^2$$

which would result from a state weighting matrix

	[1	0	• • •	0
0	0	$c^2$	• • •	0
Q =				
	0	0		0

Another possible situation is one in which we are interested in the state only through its influence on the system output

$$y = Cx$$

For example, for a system with a single output

$$y = c'x$$

a suitable performance criterion might well be

$$y^2 = x'cc'x$$

So in this case

$$Q = cc'$$

It should by now be obvious that the choice of the state weighting matrix Q depends on what the system designer is trying to achieve.

The considerations alluded to above with regard to Q apply as well to the control weighting matrix R. The term u'Ru in the performance index (9.3) is included in an attempt to limit the magnitude of the control signal u. Unless a "cost" is imposed for use of control, the design that emerges is liable to generate control signals that cannot be achieved by the actuator—the physical device that produces the control signal—and the result will be that the control signal will saturate at the maximum signal that can be produced. This is often exactly what the designer desires. In most cases, saturation of the control will produce the fastest possible response. But when saturation occurs, the closed-loop system behavior that was predicted on the basis that saturation will not occur, may be very different from the actual system behavior. A system that a linear design predicts to be stable may even be unstable when the control signal is saturated. Thus in a desire to avoid saturation and its consequences, the control signal weighting matrix is selected large enough to avoid saturation of the control signal under normal conditions of operation.

The relationship between the weighting matrices Q and R and the dynamic behavior of the closed-loop system depend of course on the matrices A and Band are quite complex. It is impractical to predict the effect on closed-loop behavior of a given pair of weighting matrices. A suitable approach for the designer would be to solve for the gain matrices G that result from a range of weighting matrices Q and R, and calculate (or simulate) the corresponding closed-loop response. The gain matrix G that produces the response closest to meeting the design objectives is the ultimate selection. With the software that is now widely available, it is a simple matter to solve for G given A, B, Q, and R. In a few hours time, the gain matrices and transient response that result for a dozen or more combinations of Q and R can be determined, and a suitable selection of G can be made.

Further comments relating to the selection of the weighting matrices will be given after the general theory is developed and illustrated by a few examples.

# 9.3 QUADRATIC INTEGRALS AND MATRIX DIFFERENTIAL EQUATIONS

When the control law (9.2) is used to control the dynamic process (9.1), the closed-loop dynamic behavior is given by

 $\dot{x} = Ax - BGx = A_c x \tag{9.5}$ 

where

 $A_c = A - BG \tag{9.6}$ 

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is the "closed-loop" dynamics matrix. In most cases considered in this text, we are interested in the case in which A, B, and G are constant matrices, but there is really no need to restrict them to be constant; in fact, the theoretical development is much easier if we do *not* assume that they are constant. Thus, we permit the closed-loop matrix  $A_c$  to vary with time. Since  $A_c$  may be time-varying we cannot write the solution to (9.5) as a matrix exponential. But the solution to (9.5) can be written in terms of the general state transition matrix introduced in Chap. 3:

$$x(\tau) = \Phi_c(\tau, t) x(t) \tag{9.7}$$

where  $\Phi_c$  is the state-transition matrix corresponding to  $A_c$ . Equation (9.7) merely states that the state at any time  $\tau$  depends linearly on the state at any other time *t*. In what follows there will be no need to have an expression for  $\Phi_c$ ; this is fortunate, because in general no simple expression is available.

Using (9.7), the performance index (9.3) can be expressed as a quadratic form in the initial state x(t). In particular

$$V = \int_{t}^{T} [x'(\tau)Qx(\tau) + x'(\tau)G'RGx(\tau)] d\tau$$
  
=  $\int_{t}^{T} x'(t)\Phi'_{c}(\tau, t)\{Q + G'RG\}\Phi_{c}(\tau, t)x(t) d\tau$  (9.8)

The initial state x(t) can be moved outside the integral to yield

$$V = x'(t)M(t, T)x(t)$$
 (9.9)

where

$$M(t, T) = \int_{t}^{T} \Phi'_{c}(\tau, t) \{Q + G'RG\} \Phi_{c}(\tau, t) d\tau$$
(9.10)

(Note that M is a symmetric matrix.)

For purposes of determining the optimum gain, i.e., the matrix G which results in the closed-loop dynamics matrix  $A_c = A - BG$  which minimizes the resulting integral (9.10), it is convenient to find a differential equation satisfied by (9.10). For this purpose, we note that V in (9.8) and (9.9) is a function of the initial time t. Thus we can write (9.8) as

$$V(t) = \int_{t}^{T} x'(\tau) L(\tau) x(\tau) d\tau \qquad (9.11)$$

where

$$L = Q + G'RG \tag{9.12}$$

(Note that L is not restricted to be constant.) Thus, by the definition of an integral

$$\left. \frac{dV}{dt} = -x'(\tau)Lx(\tau) \right|_{\tau=t} = -x'(t)Lx(t)$$
(9.13)

But, from (9.9)

$$\frac{dV}{dt} = \dot{x}'(t)M(t,T)x(t) + x'(t)\dot{M}(t,T)x(t) + x'(t)M(t,T)\dot{x}(t)$$
(9.14)

(The dot over M in (9.14) denotes differentiation with respect to t, that is,

$$\dot{M}(t, T) = \partial M(t, T) / \partial t$$

On using the closed-loop differential equation (9.5) we obtain from (9.14)

$$\frac{dV}{dt} = x'(t)[A'_c(t)M(t,T) + \dot{M}(t,T) + M(t,T)A_c(t)]x(t)$$
(9.15)

We thus have two expressions for dV/dt: one given by (9.13) and one given by (9.15). Both are quadratic forms in the initial state x(t), which is arbitrary. The only way two quadratic forms in x can be equal for any (arbitrary) x is if the matrices underlying the forms are equal. Thus we have found that the matrix M satisfies the differential equation

$$-L = A_c'M + \dot{M} + MA_c$$

or, in the more customary form

$$-M = MA_c + A'_c M + L \tag{9.16}$$

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This is an important differential equation. It appears in many forms in control theory and estimation. To make it look neater, the arguments have been omitted in (9.16). But one should not forget that

$$M = M(t, T)$$
  $A_c = A_c(t)$   $L = L(t)$ 

We have already determined the solution to (9.16) which, using (9.10), is

$$M(t, T) = \int_{t}^{T} \Phi'_{c}(\tau, t) L(t) \Phi_{c}(\tau, t) d\tau$$
(9.17)

Equation (9.16) is a first-order matrix differential equation and thus requires a single "initial condition" to pin it down completely. This condition is obtained from the integral (9.17). Clearly

$$M(T, T) = 0 (9.18)$$

is the required condition.

# 9.4 THE OPTIMUM GAIN MATRIX

When any gain matrix G is chosen to close the loop, the corresponding closed-loop performance has been shown to be given by

$$V(t) = x'(t)M(t, T)x(t)$$

where M(t, T) is the solution to (9.16), which, in terms of the matrices A, B, G,

Q, and R becomes

$$-\dot{M} = M(A - BG) + (A' - G'B')M + Q + G'RG$$
(9.19)

Our task now is to find the matrix G which makes the solution to (9.19) as small as possible. What does it mean for one matrix to be smaller than another? We are really interested in the quadratic forms resulting from these matrices, and thus we are seeking the matrix  $\hat{M}$  for which the quadratic form

$$\hat{V} = x'\hat{M}x < x'Mx$$

for any arbitrary initial state x(t) and any matrix  $M \neq \hat{M}$ .

The problem of finding an optimum gain matrix can be approached by a number of avenues. (See Note 9.1.) The approach we adopt here is to assume that a minimizing gain  $G = \hat{G}$  exists and results in an optimum (i.e., minimum  $M = \hat{M}$ ). We will then find a matrix differential equation that  $\hat{M}$  must satisfy in order for it to result in a smaller value of V than results from any other matrix.

Now the minimizing matrix  $\hat{M}$  that results from the minimizing gain  $\hat{G}$  must of course satisfy (9.19), i.e.,

$$-\hat{M} = \hat{M}(A - B\hat{G}) + (A' - \hat{G}'B')\hat{M} + Q + \hat{G}'R\hat{G}$$
(9.20)

Any nonoptimum gain matrix G and the corresponding matrix M can be expressed in terms of these matrices:

$$M = \hat{M} + N$$
$$G = \hat{G} + Z$$

Thus (9.19) becomes

$$-(\hat{M} + \dot{N}) = (\hat{M} + N)[A - B(\hat{G} + Z)] + [A' - (\hat{G}' + Z')B'](\hat{M} + N)$$
$$+ Q + (\hat{G}' + Z')R(\hat{G} + Z)$$
(9.21)

On subtracting (9.20) from (9.21) we obtain the following differential equation for N

$$-\dot{N} = NA_c + A'_c N + (\hat{G}'R - \hat{M}B)Z + Z'(R\hat{G} - B'\hat{M}) + Z'RZ \quad (9.22)$$

where  $A_c = A - BG = A - B(\hat{G} + Z)$ .

The differential equation (9.22) is exactly in the form of (9.16) with L in the latter being given by

$$L = (\hat{G}'R - \hat{M}B)Z + Z'(\hat{G}'R - \hat{M}B)' + Z'RZ$$
(9.23)

Using (9.17) we see that the solution to (9.22) is of the form

$$N(t,T) = \int_{t}^{T} \Phi_{c}'(\tau,t) L \Phi_{c}(\tau,t) d\tau \qquad (9.24)$$

Now if  $\hat{V}$  is minimum, then we must have

$$x'\hat{M}x \leq x'(\hat{M}+N)x = x'\hat{M}x + x'Nx$$

which implies that the quadratic form x'Nx must be positive-definite, or at least positive semidefinite. Now look at L as given by (9.23). If Z is sufficiently small the linear terms dominate the quadratic term Z'RZ. Thus, one can readily find values of Z which make L negative-definite, unless the linear term in (9.23) is absent altogether! Thus we conclude that for the control law  $\hat{G}$  to be optimum, we must have

$$R\hat{G} - B'\hat{M} = 0 \tag{9.25}$$

or, on the assumption that the control weighting matrix R is nonsingular

$$\hat{G} = R^{-1} B' \hat{M} \tag{9.26}$$

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This gives the optimum gain matrix in terms of the solution to the differential equation (9.20) that determines  $\hat{M}$ . When (9.26) is substituted into (9.20) the following differential equation results for  $\hat{M}$ :

$$-\hat{M} = \hat{M}A + A'\hat{M} - \hat{M}BR^{-1}B'\hat{M} + Q$$
(9.27)

This matrix differential equation, one of the most famous in the literature of modern control theory, gives the matrix  $\hat{M}$  which, using (9.26), gives the optimum gain matrix  $\hat{G}$ . (A historical discussion of the background of this equation is given in Note 9.2.)

It is noted that in addition to the linear terms  $\hat{M}A$  and  $A'\hat{M}$  in (9.27) there is also present the quadratic term  $-\hat{M}BR^{-1}B'\hat{M}$ . A scalar first-order differential equation with a linear term and a quadratic term (as well as a constant term) is known as a *Riccati equation* in the mathematical literature and the terminology was extended by Kalman[1] to the matrix case. Nowadays (9.27) is identified in the literature of optimum control as *the* Riccati equation.

Because of the presence of the quadratic term, no general formula for the solution to (9.27), analogous to the integral (9.17) for the linear equation (9.16), can be given. There are, of course, special cases—one of which is contained in Example 9E below—in which  $\hat{M}(t, T)$  can be determined analytically. But in most practical cases of interest, it is necessary to solve for  $\hat{M}(t, T)$  by some appropriate numerical method.

One obvious method of solving is the numerical integration of (9.27). Since  $\hat{M}$  is symmetric, there are k(k+1)/2 coupled, scalar equations to be integrated. It should be noted that these equations are integrated *backward* in time, because the condition that must be satisfied is

 $\hat{M}(T,T)=0$ 

and we are interested in  $\hat{M}(t, T)$  for t < T.

# 9.5 THE STEADY STATE SOLUTION

In an application in which the control interval is finite, the gain matrix G will generally be time-varying even when the matrices A, B, Q, and R are all

constant, because the solution matrix  $\hat{M}(t, T)$  of the matrix Riccati equation will not be constant. But suppose the control interval is infinite. We want a control gain G which minimizes the performance integral

$$V_{\infty} = \int_{1}^{\infty} \left( x'Qx + u'Ru \right) d\tau \tag{9.28}$$

In this case the terminal time T is infinite, so the integration (backward in time) of (9.28) will either converge to a constant matrix  $\overline{M}$  or grow without limit. If it converges to a limit, the derivative  $\hat{M}$  tends to zero. Hence for an infinite terminal time

 $V_{\infty} = x' \overline{M} x$ 

where  $\overline{M}$  satisfies the algebraic quadratic equation (sometimes called the *algebraic Riccati equation* or ARE)

$$0 = \bar{M}A + A'\bar{M} - \bar{M}BR^{-1}B'\bar{M} + Q$$
(9.29)

and the optimum gain in the steady state is given by

$$\bar{G} = R^{-1} B' \bar{M} \tag{9.30}$$

The single matrix equation of (9.29) represents a set of k(k+1)/2 coupled scalar quadratic equations. Each quadratic equation in the set in general has two solutions, so we may reasonably expect that there are 2[k(k+1)/2] = k(k+1) different (symmetric) solutions to (9.29). Are all the solutions correct? Is only one solution correct? Are there perhaps no correct solutions?

The answers to these questions are, as one might imagine, connected with the issues of stability and controllability, although from a strictly mathematical standpoint they depend on the three matrices A,  $BR^{-1}B$ , and Q and their relationships with each other. Kalman[1, 2] and others after him have addressed the issues. A complete discussion of this subject entails not only controllability, but also observability, and the more subtle concepts of *stabilizability, reconstructability*, and *detectability*, and is well beyond the scope of this book. (See Note 5.2.)

For most design applications the following facts about the solution of (9.29) will suffice:

(a) If the system is asymptotically stable, or

(b) If the system defined by the matrices (A, B) is controllable, and the system defined by (A, C) where C'C = Q, is observable,

Then the algebraic Riccati equation (ARE) has a unique, positive definite solution  $\overline{M}$  which minimizes  $V_{\infty}$  when the control law  $u = -R^{-1}B'\overline{M}x$  is used.

It should be understood that the *total* number of symmetric solutions (counting those with complex elements) is still k(k+1). The assertion of the last paragraph is that one of these solutions (and not more than one) is positivedefinite. Since the integral (9.28) is clearly positive-definite, that solution is the correct one. Let us examine the other possibilities.

It may happen that (9.29) may have no positive definite solutions. In this case, there is no control law which minimizes  $V_{\infty}$ . This must mean that  $V_{\infty}$  becomes infinite for any possible control law, and helps to explain why asymptotic stability guarantees that the ARE has a unique positive-definite solution: the control law u = 0 will result in a finite value of  $V_{\infty}$  and we would suppose that other control laws exist which can reduce  $V_{\infty}$  still further. If the system  $\dot{x} = Ax + Bu$  is not asymptotically stable, however, the control law u = 0 does not yield a finite value of  $V_{\infty}$  for any initial condition x = x(t), and it will be necessary to actively intervene with a nonzero control. This is how the idea of controllability arises, since if the system is controllable, a control law can be found which produces a closed-loop dynamics matrix  $A_c = A - BG$  with eigenvalues at arbitrary locations. Even if the system is not controllable, but merely "stabilizable," i.e., a control law can be found which can move the unstable eigenvalues to the left half-plane, a finite value of  $V_{\infty}$  can be achieved.

How does observability enter the picture? The system defined by A and B may be uncontrollable and even unstabilizable, but the matrix Q may be chosen so that those state variables corresponding to the unstabilizable portion of  $\dot{x} = Ax + Bu$  are not weighted in x'Qx. In this case, there is reason to expect that  $V_{\infty}$  can be made finite. It might at first seem strange and impractical to consider a control law which does not stabilize a system, but there are many situations in which this is entirely reasonable. The most common instance is when the state is really a "metastate" comprising both the dynamic state x and the exogenous state  $x_0$ . By hypothesis the latter cannot be controlled by the input, and it may not be asymptotically stable.

In addition to the possibility that the ARE does not have even one positive-definite solution, it is also conceivable that the ARE has more than one. Since the total number of possible solutions is finite, obviously the one we are looking for to minimize  $V_{\infty}$  is the one that yields the smallest value of  $x'M\bar{x}$ . If we could find *all* the positive-definite solutions we should surely find the proper one.

If we could only find all the solutions to (9.29) there would be no difficulty in establishing which, if any, of the solutions is the correct one. The great difficulty arises because in most practical cases, the ARE (9.29) must be solved numerically and the numerical problem is far from being easy. (See Note 9.3.) If a computer program that embodies the algorithm for solving the ARE is set to work crunching out a solution that does not exist, we should not be surprised to find it grinding away forever. So it is important to be able to find out whether the sought-after solution exists before the crunching starts.

**Example 9A Inverted pendulum** It is recalled from Example 3D that the state variables are  $x_1 = \theta$  (angular position) and  $x_2 = \dot{\theta}$  (angular velocity). The matrices defining the dynamics, as determined earlier are

$$A = \begin{bmatrix} 0 & 1 \\ \Omega^2 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
(9A.1)

A control law is sought to minimize the performance index

$$V = \int_{t}^{\infty} \left(\theta^{2} + \frac{u^{2}}{c^{2}}\right) d\tau$$
(9A.2)

where u is the angular acceleration. For this performance criterion, the weighting matrices are seen to be

 $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad R = \frac{1}{c^2}$ 

Let the performance matrix  $\hat{M}$  be given by

$$\hat{M} = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}$$

The gain matrix, in terms of the elements of  $\hat{M}$  is

$$\hat{G} = R^{-1} B' \hat{M} = c^2 [0, 1] \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} = [c^2 m_2, c^2 m_3]$$
(9A.3)

The terms needed for the matrix quadratic (9.29) are

$$\hat{M}A = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \Omega^2 & 0 \end{bmatrix} = \begin{bmatrix} m_2 \Omega^2 & m_1 \\ m_3 \Omega^2 & m_2 \end{bmatrix}$$
$$A'\hat{M} = (\hat{M}A)' = \begin{bmatrix} m_2 \Omega^2 & m_3 \Omega^2 \\ m_1 & m_2 \end{bmatrix}$$
$$\hat{M}BR^{-1}B'\hat{M} = \begin{bmatrix} c^2 m_2^2 & c^2 m_2 m_3 \\ c^2 m_2 m_3 & c^2 m_3^2 \end{bmatrix}$$
$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, the individual terms of (9.29) are

$$0 = 2m_2\Omega^2 - c^2m_2^2 + 1$$
  

$$0 = m_1 + m_3\Omega^2 - c^2m_2m_3$$
  

$$0 = 2m_2 - c^2m_3^2$$
  
(9A.4)

In this instance (9A.4) are simple enough to solve algebraically. In particular, the first equation of (9A.4) has the solution

$$m_2 = \frac{\Omega^2 \pm \sqrt{\Omega^4 + c^2}}{c^2}$$

We do not yet know which sign on the radical in  $m_2$  is correct, but we will find out shortly. From the third equation in (9A.4)

$$m_3 = \frac{1}{c}\sqrt{2m_2}$$

We note that if the lower (-) sign is used then  $m_2$  would be negative and this would make  $m_3$  imaginary. Since the matrix  $\hat{M}$  of the quadratic form  $x'(t)\hat{M}x(t)$  must be real, an imaginary number for one of its elements is unacceptable. Thus we conclude that the upper (+) sign must be used. This gives

$$m_2 = \frac{\Omega^2 + \sqrt{\Omega^4 + c^2}}{c^2} \qquad m_3 = \frac{\sqrt{2}}{c^2} [\Omega^2 + \sqrt{\Omega^4 + c^2}]^{1/2}$$



And hence the gain matrix (9A.3) has elements

$$g_1 = \Omega^2 + \sqrt{\Omega^4 + c^2}$$
  $g_2 = \sqrt{2} \left[\Omega^2 + \sqrt{\Omega^4 + c^2}\right]^{1/2}$ 

The remaining term  $m_1$  in  $\hat{M}$  is obtained from the second equation in (9A.4), but it is not needed in the control law.

The closed-loop behavior of the system is of interest. The matrix of the closed-loop system is

$$A_{c} = A - BG = \begin{bmatrix} 0 & 1 \\ \Omega^{2} & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [g_{1} \quad g_{2}]$$
$$= \begin{bmatrix} 0 & -1 \\ -\sqrt{\Omega^{4} + c^{2}} & -\sqrt{2}(\Omega^{2} + \sqrt{\Omega^{4} + c^{2}})^{1/2} \end{bmatrix}$$

And the characteristic equation is

$$s^{2} + \sqrt{2}(\Omega^{2} + \sqrt{\Omega^{4} + c^{2}})^{1/2}s + \sqrt{\Omega^{4} + c^{2}} = 0$$



Figure 9.1 Locus of closed-loop poles of controlled inverted pendulum as weighting factor is varied.

the roots of which are

where

$$s_1, s_2 = -\frac{\sqrt{2}}{2} [(\bar{\Omega}^2 + \Omega^2)^{1/2} \pm j(\bar{\Omega}^2 - \Omega^2)^{1/2}]$$
(9A.5)

$$\bar{\Omega}^2 = \sqrt{\Omega^4 + c^2}$$

The locus of closed-loop poles as the weighting factor c is varied from  $\infty$  to 0 as shown in Fig. 9.1. The following characteristics of the locus are noteworthy:

(a) As c increases, the closed-loop roots tend to asymptotes at 45° to the real axis, and move out to  $\infty$  along these asymptotes. This implies that the response time tends to zero and the damping factor tends to  $\zeta = \sqrt{2}/2 = 0.707$ . That the response time tends to zero is not surprising, since increasing c decreases the cost of control and hence makes it desirable to have a rapid response time. The asymptotic damping factor of 0.707 is entirely reasonable, since this entails good response without overshoot. But why  $\zeta \rightarrow \sqrt{2}/2$  exactly and not some other value may seem astonishing. It turns out that the root loci of second-order systems under very general conditions tend to have a damping factor of  $\zeta = \sqrt{2}/2$ . A discussion of this feature is given below.

(b) As c tends to zero, the cost of control tends to a nonzero value. If the open-loop system were stable, and it would turn out that the gains  $g_1$  and  $g_2$  would tend to zero and the open-loop system would "coast" to rest, without incurring any control cost. Since control cost is paramount, this solution would be reasonable. In the present case, however, the open-loop system is unstable, and cannot coast to rest without control. A certain amount of control is necessary to stabilize the system. But why do both closed-loop system poles tend to  $s = -\Omega$ ? One might have thought that only one closed-loop pole would tend to the stable open-loop pole at  $s = -\Omega$  and that the other would tend to the origin. The fact that the second closed-loop pole also tends to  $s = -\Omega$  is a consequence of a general result that as the control weighting becomes very large, the closed loop poles corresponding to *unstable* open loop poles tend to their *mirror images* with respect to the imaginary axis. In other words, if  $s_i = +\alpha + j\beta$  ( $\alpha \ge 0$ ) in the open-loop system, then the corresponding pole in the closed-loop system tends to  $\bar{s}_i = -\alpha + j\beta$ . This is a general property of optimum control laws, as discussed in Note 9.4.

# 9.6 DISTURBANCES AND REFERENCE INPUTS: EXOGENOUS VARIABLES

In Chap. 7 we considered a more general model for the control process of the form

$$\dot{x} = Ax + Bu + Ex_0 \tag{9.31}$$

where  $x_0$  is the exogenous vector. As in Chap. 7, we assume that  $x_0$  satisfies a known differential equation

$$\dot{x}_0 = A_0 x_0 \tag{9.32}$$

Hence the entire (meta)state satisfies the differential equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where

$$\mathbf{x} = \begin{bmatrix} x \\ \hline x_0 \end{bmatrix} \underbrace{\stackrel{\frown}{\underbrace{\uparrow}}_{k}}_{l}^{k}$$

$$\mathbf{A} = \begin{bmatrix} \underline{A} & \underline{E} \\ 0 & \underline{A}_0 \\ \downarrow & \downarrow \\ k & \downarrow \end{bmatrix}^{\frac{1}{2}} \stackrel{k}{l} \qquad \mathbf{B} = \begin{bmatrix} \underline{B} \\ 0 \end{bmatrix}$$
(9.33)

Obviously, the exogenous state  $x_0$  is not controllable; hence an appropriate performance integral would be

$$V = \int_{t}^{T} (x'Qx + u'Ru) d\tau \qquad (9.34)$$

Thus, the weighting matrix for the metastate is of the form

$$\mathbf{Q} = \begin{bmatrix} Q & \mid & 0 \\ 0 & \mid & 0 \end{bmatrix}$$

The upper limit on the integral in (9.34) is intentionally not made infinite as one might at first be tempted to do. Why? Suppose that the exogenous state does not tend to zero. It may not be possible to achieve a steady state error of zero with a control u that also goes to zero; it usually isn't possible to do so. In that case, either x'Qx doesn't go to zero, or u'Ru doesn't go to zero. In either case, the integral in (9.34) will become infinite as  $T \to \infty$ . One way of approaching this problem is to find a control  $\bar{u}$  which satisfies the requirements of zero steady state error. For  $x = \dot{x} = 0$ , the required steady state control  $\bar{u}$  must satisfy

$$B\bar{u} + Ex_0 = 0 \tag{9.35}$$

and then express the total control u as the sum of the steady state control and a "corrective" control v:

$$u = \bar{u} + v$$

In this case (9.31) becomes

$$\dot{x} = Ax + Bv$$

Now the corrective control v does tend to zero and it is proper to minimize

$$\bar{V} = \int_{t}^{\infty} \left( x'Qx + v'Rv \right) d\tau \tag{9.36}$$

There are several problems with this approach. First, there may not be a control  $\tilde{u}$  which satisfies (9.35); in other words, it may not be possible to achieve zero steady state error, but it would still be possible to minimize (9.34) for any finite time. The control *law* that is approached by the solution to (9.34) as  $T \rightarrow \infty$ , even if the limiting integral does not exist, may be just fine. Second, the control which achieves (9.35) may not be unique, hence determination of a unique v by minimizing (9.36) does not pin down  $u = \tilde{u} + v$ . And finally, minimizing the quadratic form

$$V = \int_{t}^{T} \left( x'Qx + v'Rv \right) d\tau$$

is not the same as minimizing

$$\bar{V} = \int_{t}^{T} (x'Qx + u'Ru) d\tau = \int_{t}^{T} [x'Qx + (\bar{u} + v)'R(\bar{u} + v)] d\tau$$

for a fixed  $\bar{u}$ , because of the presence of the cross terms  $v'R\bar{u}' + \bar{u}'Rv$ . If we really want to minimize V we don't want to minimize  $\bar{V}$ .

The finite time duration problem (9.3) can be solved without theoretical difficulty. Partition the performance matrix  $\hat{M}$  for the metasystem correspondingly

$$\hat{\mathbf{M}} = \begin{bmatrix} \hat{M}_1 & \hat{M}_2 \\ \hat{M}'_2 & \hat{M}_3 \end{bmatrix}$$
(9.37)

The gain matrix  $\hat{\mathbf{G}}$  for the metasystem is given by

$$\hat{\mathbf{G}} = R^{-1} \begin{bmatrix} B' & 0 \end{bmatrix} \begin{bmatrix} \hat{M}_1 & \hat{M}_2 \\ \hat{M}'_2 & \hat{M}_3 \end{bmatrix} = \begin{bmatrix} R^{-1} B' \hat{M}_1 & R^{-1} B' \hat{M}_2 \end{bmatrix}$$
(9.38)

Note that the submatrix  $\hat{M}_3$  is not needed. This is a welcome fact, as we shall soon see.

Performing the matrix multiplications required by (9.27) we obtain the differential equations for the submatrices in (9.37):

$$-\hat{M}_{1} = \hat{M}_{1}A + A'\hat{M}_{1} - \hat{M}_{1}BR^{-1}B'\hat{M}_{1} + Q$$
(9.39)

$$-\hat{M}_2 = \hat{M}_1 E + \hat{M}_2 A_0 + (A' - \hat{M}_1 B R^{-1} B') \hat{M}_2$$
(9.40)

$$-\hat{M}_3 = \hat{M}_3 A_0 + A'_0 \hat{M}_3 + \hat{M}'_2 E + E' \hat{M}_2 - \hat{M}'_2 B R^{-1} B' \hat{M}_2 \qquad (9.41)$$

Owing to the special structure of A, B, and Q, the following facts about the submatrices of  $\hat{M}$  emerge:

(a) The solution for  $\hat{M}_1$ , and hence the corresponding gain  $R^{-1}B'\hat{M}_1$ , is the same as it would have been with  $x_0$  absent from the problem, i.e., if we were designing the control law for the simple regulator problem. A steady state solution for  $\hat{M}_1$  can be obtained if the pair (A, B) is controllable, as explained above.

(b) The differential equation for  $\hat{M}_2$ , from which the gain  $R^{-1}B'\hat{M}_2$  is determined, does not depend on  $\hat{M}_3$ , and in fact is a linear equation, which can also be written

$$\hat{M}_2 = \hat{M}_1 E + \hat{M}_2 A_0 + A_c' \hat{M}_2 \tag{9.42}$$

where

$$A_c = A - BR^{-1}B'\hat{M}_1$$

is the closed-loop dynamics matrix of the regulator subsystem. A steady state solution to (9.42) generally also can be found. It must satisfy

$$0 = \bar{M}_1 E + \bar{M}_2 A_0 + A'_c \bar{M}_2 \tag{9.43}$$

We thus have the necessary gains to realize the control law

$$u = -R^{-1}B'\bar{M}_1x - R^{-1}B'\bar{M}_2x_0 \tag{9.44}$$

(c) The differential equation (9.41) for  $\hat{M}_3$  is also linear. Whether it has a steady state solution depends on  $A_0$ . If  $A_0 = 0$ , then (9.41) does not have a steady state solution. But this doesn't matter because  $\hat{M}_3$  is not used in the determination of the gain matrix.

The case of greatest interest is that in which the matrix  $A_0$  is zero. In this case the exogenous subsystem produces signals that are constant. These are the most frequently used reference signals. For this case the equations for  $\hat{M}_2$  and  $\hat{M}_3$ , as given by (9.40) and (9.41), become

$$-\hat{M}_2 = \hat{M}_1 E + A_c' \hat{M}_2 \tag{9.45}$$

and

$$-\dot{\hat{M}}_{3} = \hat{M}_{2}'E + E'\hat{M}_{2} - \hat{M}_{2}'BR^{-1}B'\hat{M}_{2}$$
(9.46)

Note that the right-hand side of (9.46) does not contain  $\hat{M}_3$  and hence

$$M_{3}(t) = M_{3}(T) + \int_{t}^{T} \left( \hat{M}_{2}'E + E'\hat{M}_{2} - \hat{M}_{2}'BR^{-1}B'\hat{M}_{2} \right) d\tau \qquad (9.47)$$

In general the integral in (9.47) goes to infinity as  $T \to \infty$ ; in other words a steady state solution to (9.46) does not exist, and any attempt to obtain such a solution, by setting  $\hat{M}_3$  to zero, will generally be erroneous. The correct relationship for  $\overline{M}_2$  is given by the solution to (9.45) with  $\hat{M}_2 = 0$ 

$$\bar{M}_2 = -(A_c')^{-1} \bar{M}_1 E \tag{9.48}$$

where  $\overline{M}_1$  is the steady state solution to (9.39), i.e., the control matrix for the regulator design. Thus the gain for the exogenous variables

$$G_0 = -R^{-1}B'(A'_c)^{-1}\bar{M}_1E = B^*E = G_0^*$$
(9.49)

where

$$B^* = -R^{-1}B'(A'_c)^{-1}\bar{M}_1$$

In Chap. 6 we considered the problem of reducing the steady state error to zero in the presence of exogenous variables. We found there that

 $G_0 = B^{\#}E$ 

where

$$B^{\#} = (CA_c^{-1}B)^{-1}CA_c^{-1} \tag{9.50}$$

By use of the optimization technique of this chapter we have also found gains for the exogenous variables. Since the matrix  $B^{\#}$  is unique for a given regulator gain matrix G, it follows that the gains given by (9.48) will reduce the steady state error to zero (for arbitrary E) only when  $B^{*}$  given by (9.49) and  $B^{\#}$  given by (9.50) are the same.

**Example 9B** Accelerometer proof mass "capture" We previously determined that the differential equations governing the displacement z of the proof mass in an accelerometer, shown in Fig. 9.2, is given by

$$\dot{x}_1 = x_2$$
 (9B.1)  
 $\dot{x}_2 = -\frac{K}{\bar{M}} x_1 - \frac{B}{\bar{M}} x_2 + a$ 

where  $K/\overline{M}$  is the acceleration due to the spring,  $B/\overline{M}$  is the acceleration due to friction, and a is the specific force (nongravitational acceleration) acting on the body. Suppose that the spring and damping forces are both absent. Then of course the proof mass would strike the end wall of the instrument after a short time. To "capture" the proof mass, i.e., to keep it from striking the end walls, a control force is generated in the typical instrument. (This can be accomplished magnetically, for example. The means of generating the force is not germane.) So instead of (9B.1), the differential equations for the proof mass, with the acceleration due to the capture force denoted by u, are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u + a$$
(9B.2)

These are just the equations of a double-integrator with an external disturbance a and a control u. For a constant acceleration a, the control acceleration must tend to -a; otherwise the proof mass will surely hit the wall. Thus, by measuring the control acceleration u that is needed to keep the proof mass from moving toward the walls, we can determine the external acceleration a. In Example 11F, using the separation principle, we will develop the design for a complete control system to capture the proof mass and provide an estimate  $\hat{a}$  of the input acceleration a. But for now, let us consider only the control problem when all the state variables, including the input acceleration a, treated as a state variable, are assumed to be measurable. (If the acceleration a could be measured, there would of course be no need for this accelerometer.)

First, consider the control problem of returning the proof mass to the origin  $(x_1 = x_2 = 0)$ in the absence of an input acceleration (a = 0). The matrices for the dynamics are

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
(9B.3)

We use a performance criterion of the form

$$V = \int_{t}^{\infty} \left( x_{1}^{2} + \frac{u^{2}}{c^{2}} \right) dt$$
 (9B.4)

The gain matrix for this control design is

$$G = R^{-1}B'M = c^{2}[0 \quad 1]\begin{bmatrix} m_{1} & m_{2} \\ m_{2} & m_{3} \end{bmatrix} = [c^{2}m_{2} & c^{2}m_{3}]$$
(9B.5]





and the components of M are given by

$$0 = -c^2 m_2^2 + 1$$
  

$$0 = m_1 - c^2 m_2 m_3$$
 (9B.6)  

$$0 = 2m_2 - c^2 m_3^2$$

the solutions to which are

$$m_1 = 2^{1/2} c^{-1/2}$$
  $m_2 = c^{-1}$   $m_3 = 2^{1/2} c^{-3/2}$  (9B.7)

(For details, see Example 9A for the inverted pendulum. The present example is a special case of Example 9A with  $\Omega^2 = 0.$ )

Using (9B.5) the gain matrix is obtained:

$$G = [c^2 m_2 \quad c^2 m_3] = [c \quad \sqrt{2c}]$$
(9B.8)

The dynamics matrix of the closed-loop system is given by

$$A_c = A - BG = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} c & \sqrt{2c} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c & -\sqrt{2c} \end{bmatrix}$$
(9B.9)

Hence the closed-loop poles are the roots of

$$|sI - A_c| = \begin{vmatrix} s & -1 \\ c & s + \sqrt{2c} \end{vmatrix} = s^2 + \sqrt{2cs} + c = 0$$

or

$$s_{1,2} = \frac{\sqrt{2}}{2}c(-1 \pm j)$$

The locus of the closed-loop poles are thus straight lines at 45 degrees to the coordinate axes and moving away from the origin as  $c \rightarrow \infty$ .

The case we really want to consider, of course, is a nonzero external acceleration. Any model for a can be used (e.g., a step, a ramp, etc.). Suppose that it is modeled as a step

$$\dot{a} = 0$$
 (9B.10)

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Adjoining this to (9B.2) gives

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{a} \end{bmatrix} = \begin{bmatrix} 0 & 1 & | & 0 \\ 0 & 0 & | & 1 \\ 0 & 0 & | & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$
(9B.11)

The matrices are in the form of (9.33) with

$$E = \begin{bmatrix} 0\\1 \end{bmatrix} \qquad A_0 = 0$$

Thus the theory developed below (9.33) applies. In particular, let

$$\hat{\mathbf{M}} = \begin{bmatrix} \hat{M}_1 & \hat{M}_2 \\ \hat{M}_2' & \hat{M}_3 \end{bmatrix} = \begin{bmatrix} m_1 & m_2 & m_4 \\ m_2 & m_3 & m_5 \\ m_4 & m_5 & m_6 \end{bmatrix}$$

Then, as already found,

$$\hat{M}_1 = \hat{M} = \begin{bmatrix} 2^{1/2}c^{-1/2} & c^{-1} \\ c^{-1} & 2^{1/2}c^{-3/2} \end{bmatrix}$$

The submatrix  $\overline{M}_2$  is found using (9.47). In this application (9.47) is

$$0 = \begin{bmatrix} 2^{1/2}c^{-1/2} & c^{-1} \\ c^{-1} & 2^{1/2}c^{-3/2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + A'_c \begin{bmatrix} m_4 \\ m_5 \end{bmatrix}$$

or, upon use of  $A_c$  given by (9B.9),

 $\bar{M}_2 = \begin{bmatrix} m_4 \\ m_5 \end{bmatrix} = -(A'_c)^{-1} \begin{bmatrix} c^{-1} \\ 2^{1/2}c^{-3/2} \end{bmatrix} = \begin{bmatrix} 0 \\ c^{-2} \end{bmatrix}$ (9B.12)

Thus the gain matrix is given by

$$\mathbf{G} = [G \ ; \ R^{-1}B'M_2]$$

The part of the gain matrix  $\overline{G}$  due to the state  $[x_1, x_2]'$  was already found in (9B.8). The additional gain due to the forcing acceleration  $a = x_3$  is

$$G_a = R^{-1} B' \bar{\mathbf{M}}_2 = c^2 [0 \quad 1] \begin{bmatrix} m_4 \\ m_5 \end{bmatrix} = 1$$

It is not in the least bit surprising that the gain for the external acceleration should turn out to be 1 exactly. In fact any other gain would be surprising: Obviously, when  $x_1$  and  $x_2$  are zero, the control acceleration u should be exactly equal in magnitude and opposite in sign to the external acceleration. Thus the control law

$$u = -g_1 x_1 - g_2 x_2 - 1 \cdot a \tag{9B.13}$$

is exactly what one would have expected to obtain.

Note that we never needed to determine the remaining term  $m_6$  of  $\overline{\mathbf{M}}$ . The differential equation for  $m_6$  is a special case of (9.45). In particular,

$$-\dot{m}_6 = 2m_5 - c^2 m_5^2 = 2c^{-2} - c^2 (c^{-2})^2 = c^{-2}$$

Thus

$$m_6(t) = m_6(T) + c^{-2}(T-t)$$

which implies that a steady state solution for  $m_6$  does not exist. This is not surprising, in view of the fact that a constant value of external acceleration demands a constant, nonzero control, and this cannot result in a finite value of the performance integral V over an infinite time interval. Nevertheless, the control law (9B.13) is eminently reasonable, provided an observer is used to estimate the unmeasured state variables  $x_3 = a$  and possibly also  $x_2 = \dot{x}_1$ . (An optimum observer design is the subject of Examples 11A and 11F.)

**Example 9C Temperature control** The temperature control considered in Example 6E and 7D can also be designed by the method of this section. Suppose we have a set of capacitances and resistances for which the dynamic model of (6E.1) becomes

$$\dot{x}_1 = -x_1 + x_2 + u$$

$$\dot{x}_2 = x_1 - 3x_2 + 2x_0$$
(9C.1)

where  $x_1$  and  $x_2$  are internal temperatures, and where  $x_0$  is the outside (ambient) temperature which may be assumed constant, i.e.,

 $\dot{x_0}$ 

$$= 0$$
 (9C.2)

From (9C.1) and (9C.2) we obtain the metasystem

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

Thus the submatrices of A and B are

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -3 \end{bmatrix} \qquad E = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad A_0 = 0 \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
(9C.3)

Before starting the calculations, take note of some features of the problem due to the physics of the process. First, it is recalled that the temperatures  $x_1$  and  $x_2$  are measured with respect to any arbitrary reference temperature. If the ambient temperature is also at the reference, i.e.,  $x_0 = 0$ , then the steady state condition is that all temperatures are equal with no heat input (u = 0). The control problem in this case is to add heat (u > 0) or remove heat (u < 0) to bring the temperature to the ambient in an optimum manner. To visualize the control problem, one might imagine that  $x_1$  and  $x_2$  are temperatures in a building which has cooled down overnight. In the morning, the indoor temperatures, having reached the nighttime ambient temperature. Thus our problem is to heat the building to raise its temperature to that of the ambient. A similar problem might be to cool a building that has reached a high daytime temperature to the ambient temperature of a pleasant summer evening.

In most climates, of course, the ambient temperature is either too hot or too cold, so that  $x_0 \neq 0$ . In the winter, heat must be added continuously (u > 0) to keep the temperature above the ambient; in the summer, heat must be removed continuously (u < 0) to keep the temperature below the ambient. Since our model of (9C.1) includes only one control variable (one heater or air conditioner) it is clear that it is not possible to keep both  $x_1$  and  $x_2$  at the reference temperature. We can control  $x_1$  or  $x_2$  or a weighted average of the two, but not both independently. The thermal model (9C.2) suggests that  $x_1$  is the temperature of the area nearest the source of heat (in a residence, perhaps downstairs) and  $x_2$  is the temperature of the area farthest from the heat (perhaps upstairs) and most prone to heat loss to the ambient environment. In the daytime we might wish to give more weight to the temperature  $x_1$ , and in the nighttime we might wish to give more weight to  $x_2$ . Thus a performance criterion of the form

$$V = \int_{t}^{T} \left[ (c_1 x_1 + c_2 x_2)^2 + k^2 u^2 \right] d\tau$$
(9C.4)

might be used, with  $c_1 \gg c_2$  in the daytime and  $c_2 \gg c_1$  in the nighttime. The state and control weighting matrices would thus be

$$Q = \begin{bmatrix} c_1^2 & c_1 c_2 \\ c_1 c_2 & c_2^2 \end{bmatrix} \qquad R = k^2$$
(9C.5)

We are now prepared to perform the required calculations. First we find the gain matrix for the case in which  $x_0 = 0$  using

$$\hat{G} = R^{-1} B' \hat{M}_1 = [g_1, g_2]$$
 (9C.6)

$$\hat{M}_1 = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}$$
(9C.7)

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satisfying (9.39). Using the data matrices of (9C.3) and (9C.5), we find that the steady state values of  $m_1$ ,  $m_2$ , and  $m_3$  satisfy

$$-2m_1 + 2m_2 - k^{-2}m_1^2 + c_1^2 = 0 (9C.8)$$

$$m_1 - 4m_2 + m_3 - k^{-2}m_1m_2 + c_1c_2 = 0 (9C.9)$$

$$2m_2 - 6m_3 - k^{-2}m_2^2 + c_2^2 = 0 (9C.10)$$

These equations are too complicated to solve other than numerically. But the numerical values are easily obtained, and from these, using (9C.6), the gains are obtained:

$$g_1 = k^{-2}m_1 \qquad g_2 = k^{-2}m_2 \tag{9C.11}$$

Numerical values for  $g_1$  and  $g_2$  for several values of  $c_1$  and  $c_2$  are given in Table 9C.1.

k	$c_1 = 1, c_2 = 0$		$c_1 = 1, c_2 = 1$		$c_1 = 0, c_2 = 1$	
	gı	<i>g</i> <sub>2</sub>	$g_1$	$g_2$	$g_1$	<i>g</i> <sub>2</sub>
1	0.4957	0.1185	0.685	0.419	0.0586	0.0603
0-1	9.120	0.7067	9.860	8.470	1.692	3.122
0^2	99.02	0.9628	99.981	98.06	10.562	66.338
0 <sup>-3</sup>	999.0	0.99615	1000.	998.0	40.855	875.44
$0^{-4}$	9999.0	0.99931	10000.	9998.0	137.46	9585.6

## Table 9C.1 Temperature control gains

The variation of gains shown in Table 9C.1 seems reasonable: as the control weighting is decreased  $(k \rightarrow 0)$  the gains get higher—as the cost of energy decreases the temperature can be brought to the ambient more rapidly. Also note that the higher gain is associated with the state variable that is weighted more heavily in the performance index—also as expected. But the gains are unequal when the states are equally weighted because the heat (or cooling) input is not distributed to  $x_1$  and  $x_2$  in the same way.

Having found  $g_1$  and  $g_2$  we can now determine the gain for the ambient temperature:

$$g_0 = B^* E$$
 (9C.12)

Where  $B^*$  is given by (9.49). Performing the required calculations we obtain the results shown in Table 9C.2.

It is of interest to compare the gain  $g_0$  given by (9C.12) with the gain  $g_0$  given by (6.48), needed for zero steady state error. For the weighting matrix

$$C = [c_1, c_2]$$
 (9C.13)

we find from (6.48) that

$$g_{0} = B^{\#}E = \frac{2c_{1}(1-g_{2})+2c_{2}(1+g_{1})}{3c_{1}+c_{2}}$$
(9C.14)  
$$= \begin{cases} \frac{2}{3}(1-g_{2}) & \text{for } c_{2} = 0\\ \frac{g_{1}-g_{2}+2}{2} & \text{for } c_{1} = c_{2}\\ 2(1+g_{1}) & \text{for } c_{1} = 0 \end{cases}$$

The results, using (9C.14) with  $g_1$  and  $g_2$  as given in Table 9C.1, are also shown in Table 9C.2 for purpose of comparison. It is evident that except for the largest values (k > 0.1) of

#### Table 9C.2 Ambient temperature gains

k	$c_1 = 1, c_2 = 0$		$c_1 = c_2 = 1$		$c_1 = 0, c_2 = 1$	
	B*E	B <sup>#</sup> E	B*E	B <sup>#</sup> E	B*E	<b>B</b> <sup>#</sup> E
1	0.2178	0.5873	0.685	1.133	0.3283	2.1172
$10^{-1}$	0.1511	0.1960	1.645	1.695	4.991	5.924
$10^{-2}$	0.02036	0.0250	1.954	1.96	23.084	23.124
10-3	0.00216	0.00253	1.995	2.0	83.65	83.71
$10^{-4}$	~0	~0	1.9995	2.0	277.0	276.9

control weighting, the gains for the ambient temperature as given by  $B^*E$  are very close to the gains given by  $B^*E$  required to reduce the steady state error precisely to zero. The differences are largely academic, because the error due to measuring temperature with any sensor of realistic quality would be greater than the errors caused by the differences between  $B^*E$  and  $B^*E$ .

**Example 9D Missile autopilot** In Chap. 6 (Example 6F) we obtained the design of a missile autopilot using a pole-placement technique. In this example we will obtain the design using the optimization methods of this chapter.

The state of the system is the difference e between the commanded and the achieved angular acceleration, the pitch rate q, and the control surface deflection  $\delta$ 

$$x = [e, q, \delta]'$$

The dynamics are given by

$$\dot{x} = Ax + Bu + Ex_0$$

where  $x_0$  is the commanded normal acceleration  $a_{NC}$ . The matrices A, B, and E are given in Example 6F.

To use the methods of this chapter it is appropriate to use a performance criterion which weights the error e and the control surface deflection  $\delta$ 

$$V = \int_{1}^{\infty} \left( e^2 + R\delta^2 \right) d\tau$$

For this performance criterion

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and R is a scalar.

The matrix quadratic equation is much too complicated to solve analytically, but it can readily be solved by a suitable numerical method. The numerical values of the elements of the gain matrix  $\bar{G} = R^{-1}B'\bar{M}$  are tabulated for a range of control weightings in Table 9D.1. Table 9D.1 also shows the closed-loop poles and the matrices  $B^*E$  and  $B^{\#}E$  which constitute the feedforward gain  $G_0$  for the reference input.

A graphical representation of the closed-loop poles, as the control weighting R is varied, is shown in Fig. 9.3. It is seen that as R becomes very large, i.e., the control surface deflection is very heavily weighted, the closed-loop poles approach the open-loop poles, as one would expect. But as the weighting on the control surface is reduced (R is decreased), the complex

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Table 9D.1 Missile autopilot design

$R$ $G_e$		$G_{a}$	$G_{\delta}$	Closed-loop poles		B*E	B <sup>#</sup> E
							1.000/00.0
E5	2.086E-3	492	5.818	-360.0,	$-46.0 \pm j6.5$	1.0755E-3	1.0786E-3
5E5	.873E-3	235	2.557	-176.4,	$-42.6 \pm j13.4$	0.5363E-3	0.5433E-3
E6	.588E-3	173	1.795	-139.7,	$-38.0 \pm j16.9$	0.4070E-3	0.4169E-3
5E6	.211E-3	085	0.784	-107.5,	$-25.3 \pm j20.4$	0.2250E-3	0.2460E-3
E7	.127E-3	063	0.545	-103.7,	$-20.0 \pm j19.9$	0.1746E-3	0.2047E-3
5E7	.274E-4	0296	0.222	-100.7,	$-10.8 \pm j17.5$	0.0881E-3	0.1457E-3
E8	.105E-4	0205	0.146	-100.3,	$-8.2 \pm j16.8$	0.0602E-3	0.1308E-3
E9	174E-5	004618	0.02883	-100.03,	$-3.18 \pm j15.8$	0.0112E-3	0.1064E-3



Figure 9.3 Locus of complex poles in missile autopilot design as weighting factor is varied.









poles move to the negative open-loop zero on the real axis, which is where we would expect it to go in view of our discussion in Note 9.4. The gain variations are illustrated graphically in Fig. 9.4.

Note that the gains  $B^*E$  and  $B^\#E$  are not equal, although they converge as the control weighting tends to zero. (Fig. 9.5.) This again is as expected in view of our earlier discussion: If the control weighting is not zero, the cost of using control requires that its steady state value be reduced from that required to maintain a steady state error of zero. The discrepancy between the feedforward gains is largest when the control weighting is largest, as expected. Since the missile is stable, the feedback gains can be reduced to zero, which is what happens when the control weighting becomes infinite. But this also reduces the feedforward gain to zero and there is no connection between the reference input (the commanded acceleration) and missile: The achieved acceleration tends to zero leaving a steady state error equal to the commanded acceleration. But it is possible to track the input acceleration perfectly, even without feedback, by use of a feedforward gain given by  $G_0 = (CA^{-1}B)^{-1}CA^{-1}E$  where A is the open-loop dynamics matrix. The numerical value of  $G_0 = 0.1064 \times 10^{-3}$  is the feedforward gain that achieves this condition.

If zero steady state tracking error is a rigid requirement, then there is no reason for not using the gain  $B^{*}E$  as given in the last column of Table 9D.1 instead of the gain given by  $B^{*}E$ . Since these are feedforward gains, they have no effect on the stability of the system.



The robustness of the design is of interest. As was the case with the design based on pole placement, as considered in Example 6F, we study the locus of roots of the return difference

$$1 + KG(sI - A)^{-1}B$$

on the assumption that the effect of a gain variation is in the overall loop gain rather than in the individual sensors.

For comparison with the pole-placement design we select the gain matrix G corresponding to a control weighting of  $R = 10^7$  which places the closed-loop poles at  $s = -20 \pm j19.9$ which is very close to the values chosen in the pole-placement design. For this value of gain we find that

$$G_0(s) = G(sI - A)^{-1}B = \frac{N(s)}{D(s)} = \frac{40.35s^2 + 4363s + 57628}{(s + 100)(s^2 + 3.33s + 248.)}$$
$$= \frac{40.35(s + 15.4)(s + 92.7)}{(s + 100)(s^2 + 3.33s + 248.)}$$

Note that the apparent zeros of the loop transmission are both in the *left half* of the s plane whereas the pole-placement design had one zero in the right half of the s plane. This means that the root-locus does not cross into the right half-plane for any value of K. Thus this design has an infinite gain margin. The actual root locus has the appearance shown in Fig. 9.6. The root locus has the same general shape as the locus of roots of the closed-loop system shown in Fig. 9.3 for various control weighting factors. (Note that Fig. 9.3 is not a standard root locus, which is defined as the locus of roots of 1 + KG(s) as K is varied.) This might seem surprising at first, but it really is quite reasonable in view of the way the gains  $G_e$ ,  $G_q$ , and  $G_\delta$  vary, as shown in Fig. 9.4. It is observed that they are nearly proportional to each other, so that varying K in the root-locus equation has nearly the same effect as varying the control weighting matrix R.

It is worth dwelling further upon the difference between the design of this section and the design obtained by pole-placement in Example 6F. The dominant poles in both cases are very nearly in the same location ( $s \approx -20 \pm j20$ ) so the transient responses of both systems would be just about the same. Yet the pole-placement design has a finite gain margin while the







Figure 9.7 Bode plots for missile autopilot. (a) Open-loop transmission; (b) Closed-loop transmission,





linear-quadratic design of this example has an infinite gain margin. A gain margin of 14 is not at all bad, but a gain margin of infinity is better! On the other hand, the present design requires feedback of the actuator state  $\delta$ . The pole-placement design intentionally eliminated this feedback path. Is it worth using an extra sensor (to measure  $\delta$ ) for the sake of raising the gain margin? In this case probably not, but in other cases it might be. The alternative to adding a sensor to measure  $\delta$  is to use an observer to estimate  $\delta$  using the measured pitch rate and normal acceleration. Use of an observer, however, also has the effect of reducing the stability margins as we shall see when our discussion of this example resumes.

The Bode plots for  $G_0(s) = G(sI - A)^{-1}B$  and  $G_c(s) = G(sI - A_c)B$  are given in Fig. 9.7. Note that the maximum phase shift of the open-loop transmission is  $-107^\circ$ , which provides a phase margin of 73°.

# 9.7 GENERAL PERFORMANCE INTEGRAL

Most problems can be formulated with a performance integral of the form (9.3) with the integrand being the sum of a quadratic form in x and a second quadratic form in u. There are cases, however, in which a cross term 2x'S'u = x'S'u + u'Sx is also present in the integral. The optimum gain for this problem can be found using the same method as was used in Sec. 9.4. Following exactly the same steps as in that section, one obtains the following relation for the optimum gain

$$\hat{G} = R^{-1}(B'\hat{M} + S) \tag{9.51}$$

where the matrix  $\hat{M}$  satisfies a matrix Riccati equation

$$-\hat{M} = \hat{M}\bar{A} + \bar{A}'\hat{M} - \hat{M}BR^{-1}B'\hat{M} + \bar{Q}$$
(9.52)

where

$$\bar{O} = O - S' R^{-1} S \tag{9.54}$$

(9.53)

The benefit of hindsight—i.e., knowing the result—makes it possible to verify it by another method. Let

 $\bar{A} = A - BR^{-1}S$ 

$$u = v - R^{-1}Sx (9.55)$$

Substitute this control into the general dynamic process, as given by (9.1) to obtain

$$\dot{x} = Ax + Bu = (A - BR^{-1}S)x + Bv = \bar{A}x + Bv$$
 (9.56)

The performance integral to be minimized is

$$V = \int_{t}^{T} (x'Qx + x'S'u + u'Sx + u'Ru) d\tau$$
(9.57)

Using (9.55) the integrand becomes

$$x'Qx + x'S'(v - R^{-1}Sx) + (v' - x'S'R^{-1})Sx + (v' - x'S'R^{-1})R(v - R^{-1}Sx)$$
  
=  $x'(Q - S'R^{-1}S)x + v'Rv$  (9.58)

Thus minimization of (9.57) for the original process is equivalent to minimization of

$$V = \int_{t}^{\infty} (x'\bar{Q}x + v'Rv) d\tau \qquad (9.59)$$

for the process

$$=\bar{A}x + Bv \tag{9.60}$$

Using the result of Sec. 9.4 the minimum value of V is obtained for

x

$$v = -\overline{G}x$$

where the gain for v is given by

$$\bar{G} = R^{-1}B'\hat{M}$$

and where  $\hat{M}$  satisfies (9.52). Thus, finally, from (9.55)

$$u = -(R^{-1}B'\hat{M} + R^{-1}S)x = -\hat{G}x$$

where  $\hat{G}$  is given by (9.51).

# 9.8 WEIGHTING OF PERFORMANCE AT TERMINAL TIME

In control processes of finite time duration, the terminal state x(T) is often as important as, or more important than, the manner in which the state is reached. Thus a more general performance criterion is

$$V = \int_{t}^{T} \left[ x'(\tau) Q x(\tau) + u'(\tau) R u(\tau) \right] d\tau + x'(T) Z x(T)$$
(9.61)

The additional quadratic form x'(T)Zx(T) may be called a *terminal penalty*—the cost of not getting to the origin at the terminal time.

The results of Secs. 9.3 and 9.4 are applicable to this problem except that the terminal condition to be used is

$$\hat{M}(T,T) = Z \tag{9.62}$$

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instead of  $\hat{M}(T, T) = 0$ .

This is seen as follows. Since

$$x(T) = \Phi_c(T, t) x(t)$$

the quadratic form x'(T)Zx(T) is also a quadratic form in the initial state:

$$x'(T)Zx(T) = x'(t)\Phi'_{c}(T,t)Z\Phi_{c}(T,t)x(t)$$
(9.63)

Thus

$$V = V(t, T) = x'(t)M(t, T)x(t)$$
(9.64)