CHAPTER

SIX

SHAPING THE DYNAMIC RESPONSE

6.1 INTRODUCTION

At last we have arrived at the point of using state-space methods for control system design. In this chapter we will develop a simple method of designing a control system for a process in which all the state variables are accessible for measurement—the method known as *pole-placement.* We will find that in a controllable system, with all the state variables accessible for measurement, it is possible to place the closed-loop poles anywhere we wish in the complex s plane. This means that we can, in principle, completely specify the closed-loop dynamic performance of the system. In principle, we can start with a sluggish open-loop system and force it to behave with alacrity; in principle, we can start with a system that has very little open-loop damping and provide any amount of damping desired. Unfortunately, however, what can be attained in principle may not be attainable in practice. Speeding the response of a sluggish system requires the use of large control signals which the actuator (or power supply) may not be capable of delivering. The consequence is generally that the actuator saturates at the largest signal that it can supply. In some instances the system behavior may be acceptable in spite of the saturation. But in other cases the effect of saturation is to make the closed-loop system unstable. It is usually not possible to alter open-loop dynamic behavior very drastically without creating practical difficulties.

Adding a great deal of damping to a system having poles near the imaginary axis is also problematic, not only because of the magnitude of the control signals needed, but also because the control system gains are very sensitive to the location of the open-loop poles. Slight changes in the open-loop pole location may cause the closed-loop system behavior to be very different from that for which it is designed.

We will first address the design of a regulator. Here the problem is to determine the gain matrix G in a linear feedback law

$$u = -Gx \tag{6.1}$$

which shapes the dynamic response of the process in the absence of disturbances and reference inputs. Afterward we shall consider the more general problem of determining the matrices G and G_0 in the linear control law

$$u = -Gx - G_0 x_0 \tag{6.2}$$

where x_0 is the vector of exogenous variables. The reason it is necessary to separate the exogenous variables from the process state x, rather than deal directly with the metastate

$$\mathbf{x} = \begin{bmatrix} x \\ \hline x_0 \end{bmatrix} \tag{6.3}$$

introduced in Chap. 5, is that in developing the theory for the design of the gain matrix, we must assume that the underlying process is *controllable.* Since the exogenous variables are not true state variables, but additional inputs that cannot be affected by the control action, they cannot be included in the state vector when using a design method that requires controllability.

The assumption that all the state variables are accessible to measurement in the regulator means that the gain matrix G in (6.1) is permitted to be any function of the state x that the design method requires. In most practical instances, however, the state variables are not all accessible for measurement. The feedback control system design for such a process must be designed to use only the measurable output of the process

$$y = Cx$$

where y is a vector of lower dimension than x. In some cases it may be possible to determine the gain matrix G_y for a control law of the form

$$u = -G_y y \tag{6.4}$$

which produces acceptable performance. But more often it is not possible to do so. It is then necessary to use a more general feedback law, of the form

$$u = -G\hat{x} \tag{6.5}$$

where $\hat{x}$ is the state of an appropriate dynamic system known as an "observer." The design of observers is the subject of Chap. 7. And in Chap. 8, we shall show that when a feedback law of the form of (6.5) is used with a properly designed observer, the dynamic properties of the overall system can be specified at will, subject to practical limitations on control magnitude and accuracy of implementation.

6.2 DESIGN OF REGULATORS FOR SINGLE-INPUT, SINGLE-OUTPUT SYSTEMS

The present section is concerned with the design of a gain matrix

$$G = g' = [g_1, g_2, \ldots, g_k] \tag{6.6}$$

for the single-input, single-output system

$$\dot{x} = Ax + Bu \tag{6.7}$$

where

$$B = b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix} \tag{6.8}$$

With the control law $u = -Gx = -g'x$ (6.7) becomes

$$\dot{x} = (A - bg')x$$

Our objective is to find the matrix $G = g'$ which places the poles of the closed-loop dynamics matrix

$$A_c = A - bg' \tag{6.9}$$

at the locations desired. We note that there are k gains $g_1, g_2, \ldots, g_k$ and k poles for a kth order system, so there are precisely as many gains as needed to specify each of the closed-loop poles.

One way of determining the gains would be to set up the characteristic polynomial for A_c:

$$|sI - A_c| = |sI - A + bg'| = s^k + \bar{a}_1 s^{k-1} + \cdots + \bar{a}_k \tag{6.10}$$

The coefficients $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_k$ of the powers of s in the characteristic polynomial will be functions of the k unknown gains. Equating these functions to the numerical values desired for $\bar{a}_1, \ldots, \bar{a}_k$ will result in k simultaneous equations the solution of which will yield the desired gains $g_1, \ldots, g_k$.

This is a perfectly valid method of determining the gain matrix g', but it entails a substantial amount of calculation when the order k of the system is higher than 3 or 4. For this reason, we would like to develop a direct formula for g in terms of the coefficients of the open-loop and closed-loop characteristic equations.

If the original system is in the companion form given in (3.90), the task is particularly easy, because

$$A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{k-1} & -a_k \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{6.11}$$

$$bg' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} [g_1, g_2, \ldots, g_k] = \begin{bmatrix} g_1 & g_2 & \cdots & g_k \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Hence

$$A_c = A - bg' = \begin{bmatrix} -a_1 - g_1 & -a_2 - g_2 & \cdots & -a_k - g_k \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

The gains $g_1, \ldots, g_k$ are simply added to the coefficients of the open-loop A matrix to give the closed-loop matrix A_c. This is also evident from the block-diagram representation of the closed-loop system as shown in Fig. 6.1. Thus for a system in the companion form of Fig. 6.1, the gain matrix elements are given by

$$a_i + g_i = \hat{a}_i \qquad i = 1, 2, \ldots, k$$

or

$$g = \hat{a} - a \tag{6.12}$$

where

$$a = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} \qquad \hat{a} = \begin{bmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_k \end{bmatrix} \tag{6.13}$$

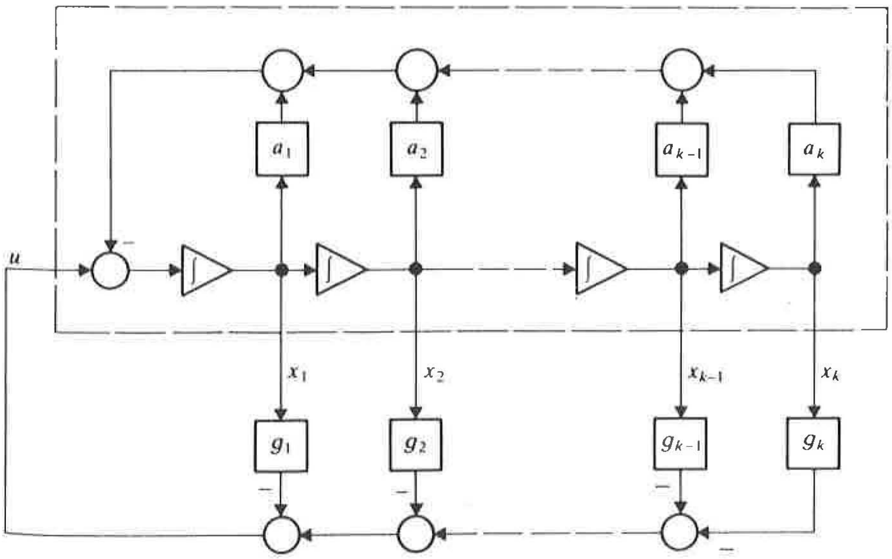

Figure 6.1 State variable feedback for system in first companion form.

are vectors formed from the coefficients of the open-loop and closed-loop characteristic equations, respectively.

The dynamics of a typical system are usually not in companion form. It is necessary to transform such a system into companion form before (6.12) can be used. Suppose that the state of the transformed system is $\bar{x}$, achieved through the transformation

$$\bar{x} = Tx \tag{6.14}$$

Then, as shown in Chap. 3,

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{b}u \tag{6.15}$$

where

$$\bar{A} = TAT^{-1} \qquad \text{and} \qquad \bar{b} = Tb$$

For the transformed system the gain matrix is

$$\bar{g} = \hat{a} - \bar{a} = \hat{a} - a \tag{6.16}$$

since $\bar{a} = a$ (the characteristic equation being invariant under a change of state variables). The desired control law in the original system is

$$u = -g'x = -g'T^{-1}\bar{x} = -\bar{g}'\bar{x} \tag{6.17}$$

From (6.17) we see that

$$\bar{g}' = g'T^{-1}$$

Thus the gain in the original system is

$$g = T'\bar{g} = T'(\hat{a} - a) \tag{6.18}$$

In words, the desired gain matrix for a general system is the difference between the coefficient vectors of the desired and actual characteristic equation, premultiplied by the inverse of the transpose of the matrix T that transforms the general system into the companion form of (3.90), the A matrix of which has the form (6.11).

The desired matrix T is obtained as the product of two matrices U and V:

$$T = VU \tag{6.19}$$

The first of these matrices transforms the original system into an intermediate system

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} \tag{6.20}$$

in the second companion form (3.107) and the second transformation U transforms the intermediate system into the first companion form.

Consider the intermediate system

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{b}u \tag{6.21}$$

with $\tilde{A}$ and $\tilde{b}$ in the form of (3.107). Then we must have

$$\tilde{A} = UAU^{-1} \qquad \text{and} \qquad \tilde{b} = Ub \tag{6.22}$$

The desired matrix U is precisely the inverse of the controllability test matrix Q of Sec. 5.4. To prove this fact, we must show that

$$U^{-1}\tilde{A} = AU^{-1} \tag{6.23}$$

or

$$Q\tilde{A} = AQ \tag{6.24}$$

Now, for a single-input system

$$Q = [b, Ab, \ldots, A^{k-1}b]$$

Thus, with $\tilde{A}$ given by (3.107), the left-hand side of (6.23) is

$$Q\tilde{A} = [b, Ab, \ldots, A^{k-1}b]\begin{bmatrix} 0 & 0 & \cdots & -a_k \\ 1 & 0 & \cdots & -a_{k-1} \\ 0 & 1 & \cdots & -a_{k-2} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -a_1 \end{bmatrix}$$

$$= [Ab, A^2b, \ldots, A^{k-1}b, -a_kb - a_{k-1}Ab - \cdots - a_kA^{k-1}b] \tag{6.25}$$

The last term in (6.25) is

$$(-a_kI - a_{k-1}A - \cdots - a_kA^{k-1})b \tag{6.26}$$

Now, by the Cayley-Hamilton theorem, (see Appendix):

$$A^k = -a_1A^{k-1} - a_2A^{k-2} - \cdots - a_kI$$

so (6.26) is A^kb. Thus the left-hand side of (6.24) as given by (6.25) is

$$Q\tilde{A} = [Ab, A^2b, \ldots, A^kb] = A[b, Ab, \ldots, A^{k-1}b] = AQ$$

which is the desired result.

If the system is not controllable, then Q^{-1} does not exist and there is no general method of transforming the original system into the intermediate system (6.21); in fact it is not possible to place the closed-loop poles anywhere one desires. Thus, controllability is an essential requirement of system design by pole placement. If the system is *stabilizable* (i.e., the uncontrollable part is asymptotically stable, as discussed in Chap. 5) a stable closed-loop system can be achieved by placing the poles of the controllable subsystem where one wishes and accepting the pole locations of the uncontrollable subsystem. In order to apply the formula of this section, it is necessary to first separate the uncontrollable subsystem from the controllable subsystem.

The control matrix $\tilde{b}$ of the intermediate system is given by

$$\tilde{b} = Ub \tag{6.27}$$

We now show that

$$\tilde{b} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{6.28}$$

Multiply (6.28) by Q to obtain

$$Q\tilde{b} = [b, Ab, \ldots, A^{k-1}b]\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = b$$

which is the same as (6.27), since $Q^{-1} = U$.

The final step is to find the matrix V that transforms the intermediate system (6.21) into the final system (6.15). We must have

$$\bar{x} = V\tilde{x} \tag{6.29}$$

For the transformation (6.28) to hold, we must have

$$\bar{A} = V\tilde{A}V^{-1}$$

or

$$V^{-1}\bar{A} = \tilde{A}V^{-1} \tag{6.30}$$

The matrix V^{-1} that satisfies (6.30) is the transpose of the upper left-hand k-by-k submatrix of the (triangular Toeplitz) matrix appearing in (3.103)

$$V^{-1} = \begin{bmatrix} 1 & a_1 & \cdots & a_{k-1} \\ 0 & 1 & \cdots & a_{k-2} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = W \tag{6.31}$$

To prove this, we note that the left-hand side of (6.30) is

$$V^{-1}\bar{A} = \begin{bmatrix} 1 & a_1 & \cdots & a_{k-1} \\ 0 & 1 & \cdots & a_{k-2} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}\begin{bmatrix} -a_1 & -a_2 & \cdots & -a_k \\ 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_k \\ 1 & a_1 & \cdots & a_{k-2} & 0 \\ 0 & 1 & \cdots & a_{k-3} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{6.32}$$

(Note that the zeros in the first row of $V^{-1}\bar{A}$ are the result of the difference of two terms $a_1 - a_1$, $a_2 - a_2$, etc.) and the right-hand side of (6.30) is

$$\tilde{A}V^{-1} = \begin{bmatrix} 0 & 0 & \cdots & -a_k \\ 1 & 0 & \cdots & -a_{k-1} \\ 0 & 1 & \cdots & -a_{k-2} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -a_k \end{bmatrix} \begin{bmatrix} 1 & a_1 & \cdots & a_{k-1} \\ 0 & 1 & \cdots & a_{k-2} \\ 0 & 0 & \cdots & a_{k-3} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_k \\ 1 & a_1 & \cdots & a_{k-2} & 0 \\ 0 & 1 & \cdots & a_{k-3} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

which is the same as (6.32). Thus (6.30) is proved.

We also need

$$\bar{b} = V\tilde{b}$$

We will show that

$$\bar{b} = \tilde{b}$$

Consider

$$\tilde{b} = V^{-1}\bar{b}$$

with

$$b = V^{-1}\bar{b} = \begin{bmatrix} 1 & a_1 & \cdots & a_{k-1} \\ 0 & 1 & \cdots & a_{k-2} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Thus $\tilde{b}$ and $\bar{b}$ are the same.

The result of this calculation is that the transformation matrix T whose transpose is needed in (6.18) is the inverse of the product of the controllability test matrix and the triangular matrix (6.31).

The above results may be summarized as follows. The desired gain matrix g, by (6.18) and (6.19), is given by

$$g = (VU)'(\hat{a} - a) \tag{6.33}$$

where

$$V = W^{-1} \qquad \text{and} \qquad U = Q^{-1}$$

Thus

$$VU = W^{-1}Q^{-1} = (QW)^{-1}$$

Hence (6.33) becomes

$$g = [(QW)']^{-1}(\hat{a} - a) \tag{6.34}$$

where Q is the controllability test matrix, W is the triangular matrix defined by (6.31), $\hat{a}$ is the vector of coefficients for the desired (closed-loop) characteristic polynomial, and a is the vector of coefficients of the open-loop system.

The basic pole-placement formula (6.34) was first stated by Bass and Gura.[1] It can be derived by other methods as discussed in Note 6.1.

Now that we have a specific formula for the gains of a controllable, single-input system that will place the poles at any desired location, several questions arise: If the closed-loop poles can be placed anywhere, where *should* they be placed? How can the technique be extended to multiple input systems? We shall address these questions and others after considering several examples.

Example 6A Instrument servo A dc motor driving an inertial load constitutes a simple instrument servo for keeping the load at a fixed position.

As shown in Chap. 2 (Example 2B), the state-space equations for the motor-driven inertia are

$$\dot{\theta} = \omega \tag{6A.1}$$

$$\dot{\omega} = -\alpha\omega + \beta u \tag{6A.2}$$

where θ is the angular position of the load, ω is the angular velocity, u is the applied voltage, and α and β are constants that depend on the physical parameters of the motor and load:

$$\alpha = -K^2/JR \qquad \beta = K/JR$$

If the desired position θ_r is a constant then we can define the servo error

$$e = \theta - \theta_r$$

Then

$$\dot{e} = \dot{\theta} - \dot{\theta}_r = \omega \qquad (\theta_r = \text{const}) \tag{6A.3}$$

and (6A.3) replaces (6A.1) to give

$$\begin{bmatrix} \dot{e} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix} \begin{bmatrix} e \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ \beta \end{bmatrix} u \tag{6A.4}$$

The angular position measurement can be instrumented by a potentiometer on the motor shaft and the angular velocity by a tachometer. Thus, the closed-loop system would have the configuration illustrated in Fig. 6.2. Note that the position gain is shown multiplying the negative of the system error which in turn is added to the control signal. This is consistent with the convention normally used for servos, wherein the position gain multiplies the difference $\theta_r - \theta$ between the reference and the actual positions. The quantity e defined above (6A.3) is the negative of the system error as normally defined in elementary texts.

The characteristic polynomial of the system is

$$|sI - A| = \begin{vmatrix} s & -1 \\ 0 & s+\alpha \end{vmatrix} = s^2 + \alpha s$$

Thus

$$a = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$$

The controllability test matrix Q and the matrix W are given respectively by

$$Q = [b, Ab] = \begin{bmatrix} 0 & \beta \\ \beta & -\alpha\beta \end{bmatrix} \qquad W = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$$

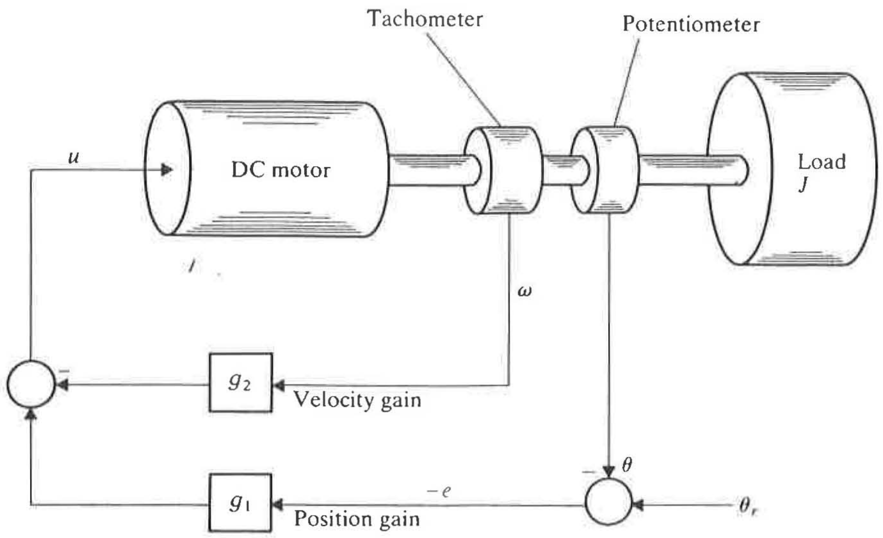

Figure 6.2 Implementation of an instrument servo.

Thus

$$QW = \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix} = (QW)'$$

and

$$[(QW)']^{-1} = \begin{bmatrix} 0 & 1/\beta \\ 1/\beta & 0 \end{bmatrix}$$

Thus the desired gain matrix, by the Bass-Gura formula (6.34), is

$$g = \begin{bmatrix} 0 & 1/\beta \\ 1/\beta & 0 \end{bmatrix} \begin{bmatrix} \bar{a}_1 - \alpha \\ \bar{a}_2 \end{bmatrix} = \begin{bmatrix} \bar{a}_2/\beta \\ (\bar{a}_1 - \alpha)/\beta \end{bmatrix} \tag{6A.5}$$

where $\bar{a}_1$ and $\bar{a}_2$ are the coefficients of the desired characteristic polynomial.

While the above calculation illustrates the general procedure, the gains could have been more easily computed directly. For a control law of the form

$$u = -g_1 e - g_2 \omega$$

(6A.4) becomes

$$\dot{e} = \omega$$

$$\dot{\omega} = -g_1 \beta e - (\alpha + \beta g_2)\omega$$

which has the closed-loop matrix

$$A_c = \begin{bmatrix} 0 & 1 \\ -g_1\beta & -(\alpha + g_2\beta) \end{bmatrix}$$

with the characteristic equation

$$|sI - A_c| = s^2 + (\alpha + g_2\beta)s + g_1\beta$$

Thus

$$\bar{a}_1 = \alpha + g_2\beta \qquad \bar{a}_2 = g_1\beta$$

or

$$g_1 = \bar{a}_2/\beta \qquad g_2 = (\bar{a}_1 - \alpha)/\beta$$

which is the same as (6A.5).

Note that the position and velocity gains g_1 and g_2, respectively, are proportional to the amounts we wish to move the coefficients from their open-loop positions. The position gain g_1 is necessary to produce a stable system: $\bar{a}_2 > 0$. But if the designer is willing to settle for $a_1 = \alpha$, i.e., to accept the open-loop damping, then the gain g_2 can be zero. This of course eliminates the need for a tachometer and reduces the hardware cost of the system. It is also possible to alter the system damping without the use of a tachometer, by using an estimate $\hat{\omega}$ of the angular velocity ω. This estimate is obtained by means of an observer as discussed in Chap. 7.

Example 6B Stabilization of an inverted pendulum An inverted pendulum can readily be stabilized by a closed-loop feedback system, just as a person of moderate dexterity can do it.

A possible control system implementation is shown in Fig. 6.3, for a pendulum constrained to rotate about a shaft at its bottom point. The actuator is a dc motor. The angular position of the pendulum, being equal to the position of the shaft to which it is attached, is measured by means of a potentiometer. The angular velocity in this case can be measured by a "velocity pick-off" at the top of the pendulum. Such a device could consist of a coil of wire

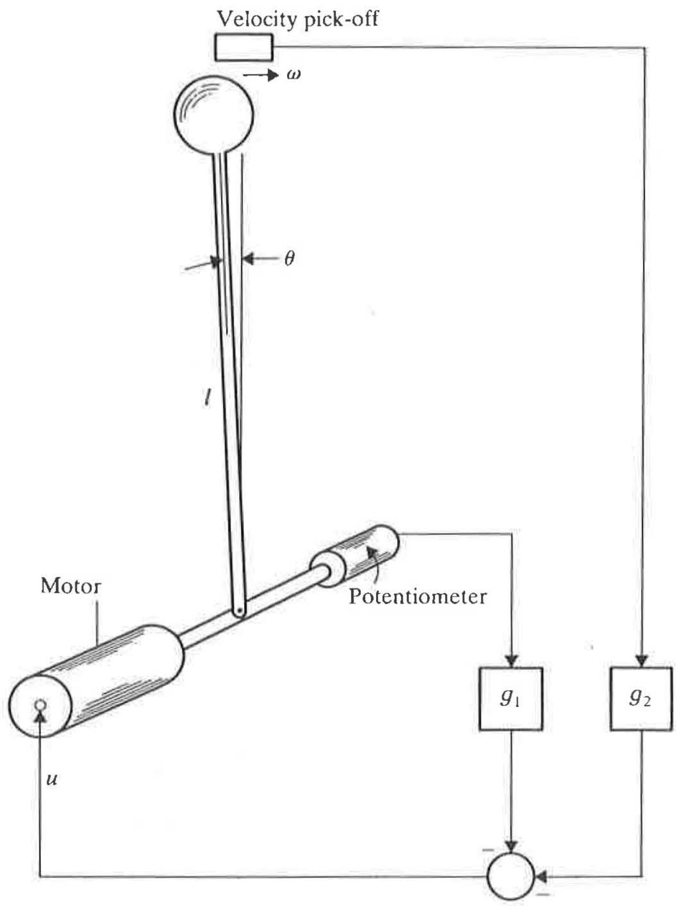

Figure 6.3 Implementation of system to stabilize inverted pendulum.

in a magnetic field created by a small permanent magnet in the pendulum bob. The induced voltage in the coil is proportional to the linear velocity of the bob as it passes the coil. And since the bob is at a fixed distance from the pivot point the linear velocity is proportional to the angular velocity. The angular velocity could of course also be measured by means of a tachometer on the dc motor shaft.

As determined in Prob. 2.2, the dynamic equations governing the inverted pendulum in which the point of attachment does not translate is given by

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= \Omega^2\theta - \alpha\omega + \beta u \end{aligned} \qquad (6B.1)$$

where α and β are given in Example 6A, with the inertia J being the total reflected inertia:

$$J = J_m + ml^2$$

where m is the pendulum bob mass and l is the distance of the bob from the pivot. The natural frequency Ω is given by

$$\Omega^2 = \frac{mgl}{J + ml^2} = \frac{g}{l + J/ml}$$

(Note that the motor inertia J_m affects the natural frequency.)

Since the linearization is valid only when the pendulum is nearly vertical, we shall assume that the control objective is to maintain $\theta = 0$. Thus we have a simple regulator problem.

The matrices A and b for this problem are

$$A = \begin{bmatrix} 0 & 1 \\ \Omega^2 & -\alpha \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ \beta \end{bmatrix}$$

The open-loop characteristic polynomial is

$$|sI - A| = \begin{vmatrix} s & -1 \\ -\Omega^2 & s + \alpha \end{vmatrix} = s^2 + \alpha s - \Omega^2$$

Thus

$$\begin{aligned} a_1 &= \alpha \\ a_2 &= -\Omega^2 \end{aligned}$$

The open-loop system is unstable, of course.

The controllability test matrix and the W matrix are given respectively by

$$Q = \begin{bmatrix} 0 & \beta \\ \beta & -\alpha\beta \end{bmatrix} \qquad W = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$$

(which are the same as they were for the instrument servo). And

$$[(QW)']^{-1} = \begin{bmatrix} 0 & 1/\beta \\ 1/\beta & 0 \end{bmatrix}$$

Thus the gain matrix required for pole placement using (6.34), is

$$g = \begin{bmatrix} 0 & 1/\beta \\ 1/\beta & 0 \end{bmatrix} \begin{bmatrix} (\bar{a}_1 - \alpha) \\ \bar{a}_2 + \Omega^2 \end{bmatrix} = \begin{bmatrix} (\bar{a}_2 + \Omega^2)/\beta \\ (\bar{a}_1 - \alpha)/\beta \end{bmatrix}$$

Example 6C Control of spring-coupled masses The dynamics of a pair of spring-coupled masses, shown in Fig. 3.7(a), were shown in Example 3I to have the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -K/\bar{M} & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The system has the characteristic polynomial

$$D(s) = s^4 + (K/\bar{M})s^2$$

Hence $$a_1 = a_3 = a_4 = 0, \qquad a_2 = K/\bar{M}.$$

The controllability test and W matrices are given, respectively, by

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -K/\bar{M} \\ 1 & 0 & -K/\bar{M} & 0 \end{bmatrix} \qquad W = \begin{bmatrix} 1 & 0 & K/\bar{M} & 0 \\ 0 & 1 & 0 & K/\bar{M} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6C.1}$$

Multiplying we find that

$$QW = (QW)' = (QW)^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{6C.2}$$

(This rather simple result is not really as surprising as it may at first seem. Note that A is in the first companion form but using the right-to-left numbering convention. If the left-to-right numbering convention were used the A matrix would already be in the companion form of (6.11) and would not require transformation. The transformation matrix T given by (6C.2) has the effect of changing the state variable numbering order from left-to-right to right-to-left, and vice versa.)

The gain matrix g is thus given by

$$g = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 - K/\bar{M} \\ \bar{a}_3 \\ \bar{a}_4 \end{bmatrix} = \begin{bmatrix} \bar{a}_4 \\ \bar{a}_3 \\ \bar{a}_2 - K/\bar{M} \\ \bar{a}_1 \end{bmatrix}$$

A suitable pole "constellation" for the closed-loop process might be a Butterworth pattern as discussed in Sec. 6.5. To achieve this pattern the characteristic polynomial should be of the form

$$\bar{D}(s) = s^4 + (1+\sqrt{3})\Omega s^3 + (2+\sqrt{3})\Omega^2 s^2 + (1+\sqrt{3})\Omega^3 s + \Omega^4$$

Thus

$$\bar{a}_1 = (1+\sqrt{3})\Omega$$
$$\bar{a}_2 = (2+\sqrt{3})\Omega^2$$
$$\bar{a}_3 = (1+\sqrt{3})\Omega^3$$
$$\bar{a}_4 = \Omega^4$$

Thus the gain matrix g is given by

$$g = \begin{bmatrix} \Omega^4 \\ (1+\sqrt{3})\Omega^3 \\ (2+\sqrt{3})\Omega^2 - K/\bar{M} \\ (1+\sqrt{3})\Omega \end{bmatrix}$$

6.3 MULTIPLE-INPUT SYSTEMS

If the dynamic system under consideration

$$\dot{x} = Ax + Bu$$

has more than one input, that is, B has more than one column, then the gain matrix G in the control law

$$u = -Gx$$

has more than one row. Since each row of G furnishes k gains that can be adjusted, it is clear that in a controllable system there will be more gains available than are needed to place all of the closed-loop poles. This is a benefit: the designer has more flexibility in the design than in the case of a single-input system; it is possible to specify all the closed-loop poles and still be able to satisfy other requirements. How should these other requirements be specified? The answer to this question may well depend on the circumstances of the particular application. One possibility might be to set some of the gains to zero. For example, it is sometimes possible to place the closed-loop poles at locations desired with a gain matrix which has a column of zeros. This means that the state variable corresponding to that column is not needed in the generation of any of the control signals in the vector u, and hence there is no need to measure (or estimate) that state variable. This simplifies the resulting control system structure. If all the state variables, except those corresponding to columns of zeros in the gain matrix, are accessible for measurement then there is no need for an observer to estimate the state variables that cannot be measured. A very simple and robust control system is the result.

Another possible method of selecting a particular structure for the gain matrix is to make each control variable depend on a different group of state variables which are physically more closely related to that control variable than to the other control variables.

Still another possibility arises in systems which have a certain degree of structural symmetry and in which it is desired to retain the symmetry in the closed-loop system by an appropriate feedback structure.

The following example illustrates one method of selecting the gain matrix.

Example 6D Distillation column For the distillation column of Example 4A, having the block-diagram of Fig. 4.2, we saw in Example 5G that both inputs are needed in order for the system to be controllable, because there are redundant poles at the origin (due to the integrators) from either Δu_1 or Δu_2. If there were only one integrator present, it is easy to see that the system would be controllable from Δu_1 alone. This suggests a gain structure in which Δu_1 depends on x_1, x_2, and x_3, and Δu_2 depends on x_4. This gives four adjustable gains for the closed-loop fourth-order system and we would expect to be able to locate the closed-loop poles at whatever locations are desired.

Thus we use a gain matrix of the form

$$G = \begin{bmatrix} g_1 & g_2 & g_3 & 0 \\ 0 & 0 & 0 & g_4 \end{bmatrix} \tag{6D.1}$$

With the A and B matrices as given by (2G.5) it is found that the closed-loop dynamics matrix is

$$A_c = A - BG = \begin{bmatrix} a_{11} - b_{11}g_1 & -b_{11}g_2 & -b_{11}g_3 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & a_{32} & 0 & -b_{32}g_4 \\ 0 & 0 & 0 & -b_{42}g_4 \end{bmatrix}$$