METR4202 -- Robotics
Tutorial 4 – Week 4: Solutions

Solutions updated by Chris, Jeevan, Russell. Thank you!

Reading
Please read/review chapter 8 & 9 of Robotics, Vision and Control.

Review: Forward Kinematics of a two-link planar manipulator

$$x = a_1 cos\theta_1 + a_2 \cos(\theta_1 + \theta_2)$$
$$y = a_1 \sin\theta_1 + a_2 \sin(\theta_1 + \theta_2)$$

Questions

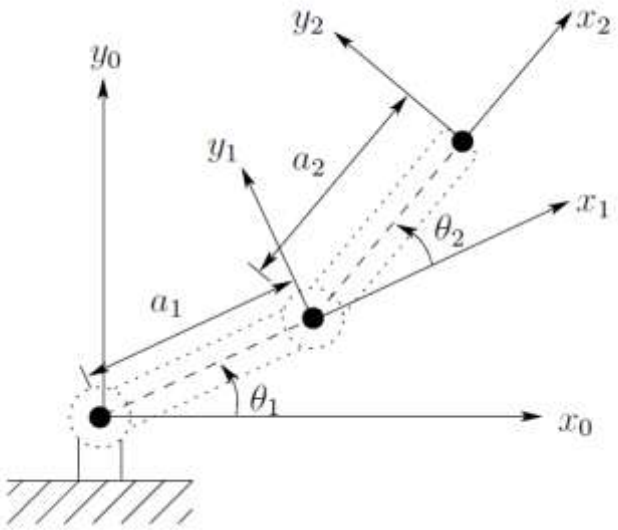

Figure 1: Two-link planar manipulator

1.

a.) Using the two-link planar manipulator from tutorial 3, calculate the jacobian needed to relate the joint velocities to the tool-point velocities.

$$J = \begin{bmatrix} -l_1 \sin(\theta_1) - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

b.) Similarly, calculate the inverse jacobian needed to relate the tool-point velocities to the joint velocities.

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \frac{1}{L_1 L_2 s_2} \begin{bmatrix} L_2 c_{12} & L_2 s_{12} \\ -L_1 c_1 - L_2 c_{12} & -L_1 s_1 - L_2 s_{12} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

2.

a.) Using the Jacobian found in Q1a, calculate the tool point linear velocity if joint 1 is rotating at 1 rad/s and joint 2 is rotating at 3 rad/s ($a_1 = 2$, $a_2 = 3$ $\theta_1 = 167.028^o$, $\theta_2 = -156.44^o$).

$$\vec{v} = J\vec{\dot{q}}$$

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}-l_1\sin(\theta_1) - l_2\sin(\theta_1+\theta_2) & -l_2\sin(\theta_1+\theta_2)\\ l_1\cos(\theta_1) + l_2\cos(\theta_1+\theta_2) & l_2\cos(\theta_1+\theta_2)\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix}$$

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}-1.0000 & -0.5512\\ 1.0000 & 2.9489\end{bmatrix}\begin{bmatrix}1\\ 3\end{bmatrix}$$

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}-2.654\\ 9.847\end{bmatrix} m/s$$

b.) Calculate the resulting joint torques τ, given a force F = (30, -20) is applied to the end effector tool point.

$$\tau = J^T F$$

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}-l_1\sin(\theta_1) - l_2\sin(\theta_1+\theta_2) & l_1\cos(\theta_1) + l_2\cos(\theta_1+\theta_2)\\ -l_2\sin(\theta_1+\theta_2) & l_2\cos(\theta_1+\theta_2)\end{bmatrix}\begin{bmatrix}F_1\\ F_2\end{bmatrix}$$

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}-1.0000 & 1.0000\\ -0.5512 & 2.9489\end{bmatrix}\begin{bmatrix}30\\ -20\end{bmatrix}$$

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}-50.0\\ -75.1\end{bmatrix} Nm$$

3. (See also p. 209 of Spong, *Robot Modeling and Control* [p. 17 of attached PDF] or Ex 13.13 (p.637) of LaValle, *Planning Algorithms* [p.772 of Ch. 13 of the online PDF], or p. 110 of Asada and Slotine, *Robot Analysis and Control*)

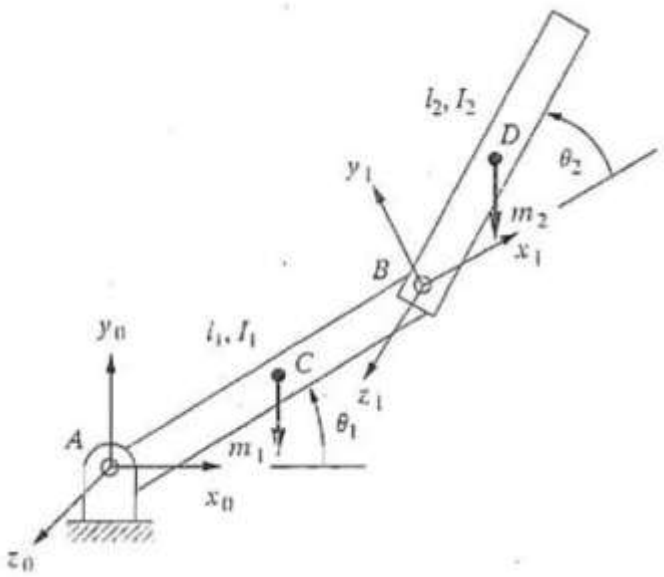

Figure 2: Two-link revolute joint arm.

a.) With respect to figure 2 above, derive the equations of motion for the two-degree-of-freedom robot arm using the Lagrangian method.

$$\tau_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i}$$

Step 1: Calculate the velocities at the center of mass of link 2.

$$x_D = l_1C_1 + 0.5l_2C_{12} \Rightarrow \dot{x_D} = -l_1S_1\dot{\theta}_1 - 0.5l_2S_{12}(\dot{\theta}_1 + \dot{\theta}_2)$$
$$y_d = l_1S_1 + 0.5l_2S_{12} \Rightarrow \dot{y_d} = l_1C_1\dot{\theta}_1 + 0.5l_2C_{12}(\dot{\theta}_1 + \dot{\theta}_2)$$

Total Velocity :

$$v_D{}^2 = \dot{x_D^2} + \dot{y_D^2}$$
$$= \dot{\theta}_1(l_1^2 + 0.25l_2^2 + l_1l_2C_2) + \dot{\theta}_2^2(0.25l_2^2) + \dot{\theta}_1\dot{\theta}_2(0.5l_2^2 + l_1l_2C_2)$$

Step 2: Calculate total Kinetic energy.

$$K = K_1 + K_2$$
$$= \left[\frac{1}{2}I_A\dot{\theta}_1^2\right] + \left[\frac{1}{2}I_D(\dot{\theta}_1 + \dot{\theta}_2)^2 + \frac{1}{2}m_2v_d^2\right]$$

Substitute the total velocity into the Kinetic energy.

$$K = \dot{\theta}^2\left(\frac{1}{6}m_1l_1^2 + \frac{1}{6}m_2l_2^2 + \frac{1}{2}m_2l_1^2 + \frac{1}{2}m_2l_1l_2C_2\right) + \dot{\theta}_2^2\left(\frac{1}{6}m_2; l_2^2\right)$$
$$+ \dot{\theta}_1\dot{\theta}_2\left(\frac{1}{3}m_2l_2^2 + \frac{1}{2}m_2l_1l_2C_2\right)$$

Step 3: Calculate the total Potential energy of the system.

$$P = \frac{m_1gl_1}{2}S_1 + m_2g(l_1S_1 + \frac{l_2}{2}S_{12})$$

The Lagrangian for the two-link robot arm is:

$$L = K - P$$
$$= \dot{\theta}^2\left(\frac{1}{6}m_1l_1^2 + \frac{1}{6}m_2l_2^2 + \frac{1}{2}m_2l_1^2 + \frac{1}{2}m_2l_1l_2C_2\right) + \dot{\theta}_2^2\left(\frac{1}{6}m_2; l_2^2\right)$$
$$+ \dot{\theta}_1\dot{\theta}_2\left(\frac{1}{3}m_2l_2^2 + \frac{1}{2}m_2l_1l_2C_2\right) - \frac{m_1gl_1}{2}S_1 - m_2g(l_1S_1 + \frac{l_2}{2}S_{12})$$

Step 4: Calculate the derivatives of the Lagrangian to determine the torque equations for the two-link robot arm: **Recall Chain rule expansion:**

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\frac{d}{dt} = \frac{\partial}{\partial\theta_2}\frac{d\theta_2}{dt} + \frac{\partial}{\partial\dot{\theta_1}}\frac{d\dot{\theta}_1}{dt}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta_1}}\right)$$
$$= \left(\frac{1}{3}m_1l_1^2 + m_2l_1^2 + \frac{1}{3}m_2l_2^2 + m_2l_1l_2C_2\right)\ddot{\theta_1}$$
$$+ \left(\frac{1}{3}m_2l_2^2 + \frac{1}{2}m_2l_1l_2C_2\right)\ddot{\theta_2} - (m_2l_1l_2S_2)\dot{\theta}_1\dot{\theta}_2 - \left(\frac{1}{2}m_2l_1l_2S_2\right)\dot{\theta}_2^2$$

$$\frac{\partial L}{\partial\theta_1} = \left(\frac{1}{2}m_1 + m_2\right)gl_1C_1 + \frac{1}{2}m_2gl_2C_{12}$$

$$\tau_1 = \left(\tfrac{1}{3}m_1l_1^2 + m_2l_1^2 + \tfrac{1}{3}m_2l_2^2 + m_2l_1l_2C_2\right)\ddot{\theta_1} + \left(\tfrac{1}{3}m_2l_2^2 + \tfrac{1}{2}m_2l_1l_2C_2\right)\ddot{\theta_2} -$$
$$(m_2l_1l_2S_2)\dot{\theta}_1\dot{\theta}_2 - \left(\tfrac{1}{2}m_2l_1l_2S_2\right)\dot{\theta}_2^2 + \left(\tfrac{1}{2}m_1 + m_2\right)gl_1C_1 + \tfrac{1}{2}m_2gl_2C_{12}$$

$$\frac{d}{dt} = \frac{\partial}{\partial\theta_2}\frac{d\theta_2}{dt} + \frac{\partial}{\partial\dot{\theta_2}}\frac{d\dot{\theta}_2}{dt}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta_2}}\right)$$
$$= \left(\frac{1}{3}m_2l_2^2 + \frac{1}{2}m_2l_1l_2C_2\right)\ddot{\theta_1} + (\frac{1}{3}m_2l_2^2)\ddot{\theta_2} - \left(\frac{1}{2}m_2l_1l_2S_2\right)\dot{\theta}_1^2$$
$$- (m_2l_1l_2S_2)\dot{\theta}_1\dot{\theta}_2$$

$$\frac{\partial L}{\partial\theta_2} = (m_2l_1l_2S_2)\dot{\theta}_1\dot{\theta}_2 + \frac{1}{2}m_2gl_2C_{12}$$

$$\tau_2 = \left(\frac{1}{3}m_2l_2^2 + \frac{1}{2}m_2l_1l_2C_2\right)\ddot{\theta_1} + \left(\frac{1}{3}m_2l_2^2\right)\ddot{\theta_2} - \left(\frac{1}{2}m_2l_1l_2S_2\right)\dot{\theta}_1^2 + \frac{1}{2}m_2gl_2C_{12}$$

Challenge Question:

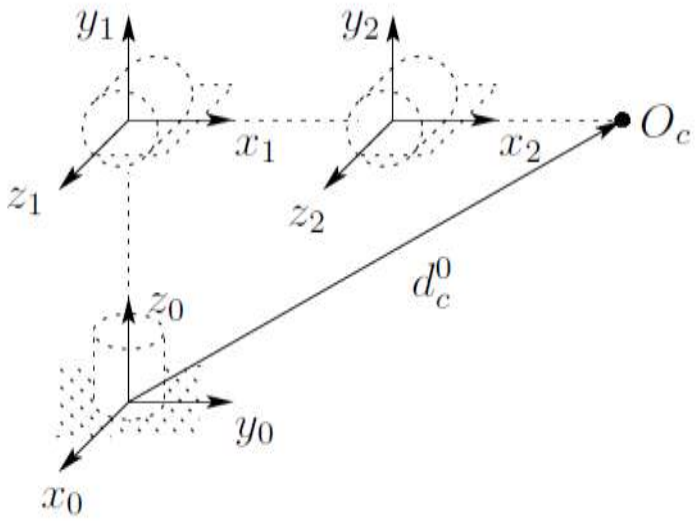

Figure 3: Elbow Manipulator

a.) List the DH parameters for this arm, clearly indication which parameters are the joint variables (L_1 = 3m. L_2 = 2m, L_3 = 1m).

b.) Find the inverse Kinematic equations for the arm to derive the joint values from tool point position.

c.) Given that the tool point is at $(1.0m, 0.2m, 0.5m)^T$, use the inverse kinematic equations to find the joint values.

d.) Find the manipulator Jacobian, J, that relates the joint velocities to the tool point velocity.

Chapter 9

DYNAMICS

This chapter deals with the dynamics of robot manipulators. Whereas the kinematic equations describe the motion of the robot without consideration of the forces and moments producing the motion, the dynamic equations explicitly describe the relationship between force and motion. The equations of motion are important to consider in the design of robots, as well as in simulation and animation, and in the design of control algorithms. We introduce the so-called **Euler-Lagrange equations**, which describe the evolution of a mechanical system subject to **holonomic constraints** (this term is defined later on). To motivate the Euler-Lagrange approach we begin with a simple derivation of these equations from Newton's Second Law for a one-degree-of-freedom system. We then derive the Euler-Lagrange equations from the **Principle of Virtual Work** in the general case.

In order to determine the Euler-Lagrange equations in a specific situation, one has to form the **Lagrangian** of the system, which is the difference between the **kinetic energy** and the **potential energy**; we show how to do this in several commonly encountered situations. We then derive the dynamic equations of several example robotic manipulators, including a two-link cartesian robot, a two-link planar robot, and a two-link robot with remotely driven joints.

The Euler-Lagrange equations have several very important properties that can be exploited to design and analyze feedback control algorithms. Among these are explicit bounds on the inertia matrix, linearity in the inertia parameters, and the so-called skew symmetry and passivity properties. We discuss these properties in Section 9.5.

This chapter is concluded with a derivation of an alternate the formulation of the dynamical equations of a robot, known as the **Newton-Euler formulation** which is a recursive formulation of the dynamic equations that is often used for numerical calculation.

9.1 The Euler-Lagrange Equations

In this section we derive a general set of differential equations that describe the time evolution of mechanical systems subjected to holonomic constraints, when the constraint forces satisfy the principle of virtual work. These are called the **Euler-Lagrange equations** of

motion. Note that there are at least two distinct ways of deriving these equations. The method presented here is based on the method of virtual displacements; but it is also possible to derive the same equations based on Hamilton's principle of least action [**?**].

9.1.1 One Dimensional System

To motivate the subsequent derivation, we show first how the Euler-Lagrange equations can be derived from Newton's Second Law for a single degree of freedom system consisting of a particle of constant mass m, constrained to move in the y-direction, and subject to a force f and the gravitational force mg, as shown in Figure 9.1. By Newton's Second law,

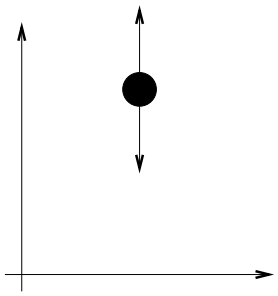

Figure 9.1: One Degree of Freedom System

$F = ma$, the equation of motion of the particle is

$$m\ddot{y} \quad = \quad f - mg \tag{9.1}$$

Notice that the left hand side of Equation (9.1) can be written as

$$m\ddot{y} = \frac{d}{dt}(m\dot{y}) = \frac{d}{dt}\frac{\partial}{\partial \dot{y}}\left(\frac{1}{2}m\dot{y}^2\right) = \frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{y}} \tag{9.2}$$

where $\mathcal{K} = \frac{1}{2}m\dot{y}^2$ is the **kinetic energy**. We use the partial derivative notation in the above expression to be consistent with systems considered later when the kinetic energy will be a function of several variables. Likewise we can express the gravitational force in Equation (9.1) as

$$mg \quad = \quad \frac{\partial}{\partial y}(mgy) \quad = \quad \frac{\partial \mathcal{P}}{\partial y} \tag{9.3}$$

where $\mathcal{P} = mgy$ is the **potential energy due to gravity**. If we define

$$\mathcal{L} \quad = \quad \mathcal{K} - \mathcal{P} \quad = \quad \frac{1}{2}m\dot{y}^2 - mgy \tag{9.4}$$

and note that

$$\frac{\partial \mathcal{L}}{\partial \dot{y}} \quad = \quad \frac{\partial \mathcal{K}}{\partial \dot{y}} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial y} \quad = \quad -\frac{\partial \mathcal{P}}{\partial y}$$

then we can write Equation (9.1) as

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{\partial \mathcal{L}}{\partial y} = f. \tag{9.5}$$

The function $\mathcal{L}$, which is the difference of the kinetic and potential energy, is called the **Lagrangian** of the system, and Equation (9.5) is called the **Euler-Lagrange Equation**. The Euler-Lagrange equations provide a formulation of the dynamic equations of motion equivalent to those derived using Newton's Second Law. However, as we shall see, the Lagrangian approach is advantageous for more complex systems such as multi-link robots.

Example: 9.1 *Single-Link Manipulator*

Consider the single-link robot arm shown in Figure 9.2, consisting of a rigid link coupled

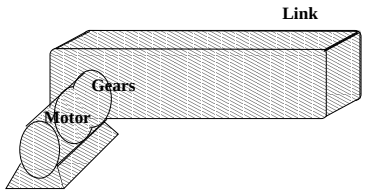

Figure 9.2: Single-Link Robot.

through a gear train to a DC-motor. Let θ_ℓ and θ_m denote the angles of the link and motor shaft, respectively. Then $\theta_m = r\theta_\ell$ where $r:1$ is the gear ratio. The algebraic relation between the link and motor shaft angles means that the system has only one degree-of-freedom and we can therefore write the equations of motion using either θ_m or θ_ℓ. In terms of θ_ℓ, the kinetic energy of the system is given by

$$\begin{aligned} K &= \frac{1}{2}J_m\dot{\theta}_m^2 + \frac{1}{2}J_\ell\dot{\theta}_\ell^2 \\ &= \frac{1}{2}(r^2J_m + J_\ell)\dot{\theta}_\ell^2 \end{aligned} \tag{9.6}$$

where J_m, J_ℓ are the rotational inertias of the motor and link, respectively. The potential energy is given as

$$P = Mg\ell(1-\cos\theta_\ell) \tag{9.7}$$

where M is the total mass of the link and ℓ is the distance from the joint axis to the link center of mass. Defining $J = r^2J_m + J_\ell$, the Lagrangian $\mathcal{L}$ is given by

$$\mathcal{L} = \frac{1}{2}J\dot{\theta}_\ell^2 - Mg\ell(1-\cos\theta_\ell). \tag{9.8}$$

Substituting this expression into the Euler-Lagrange equations yields the equation of motion

$$J\ddot{\theta}_\ell + Mg\ell\sin\theta_\ell = \tau_\ell. \tag{9.9}$$

The generalized force τ_ℓ consists of the motor torque $u = r\tau_m$, reflected to the link and (nonconservative) damping torques $B_m\dot{\theta}_m$, and $B_\ell, \dot{\theta}_\ell$. Reflecting motor damping to the link yields

$$\tau \quad = \quad u - B\dot{\theta}_\ell.$$

where $B = rB_m + B_\ell$. Therefore the complete expression for the dynamics of this system is

$$J\ddot{\theta}_\ell + B\dot{\theta}_\ell + Mg\ell \sin\theta_\ell \quad = \quad u. \tag{9.10}$$

In general, for any system of the type considered, an application of the Euler-Lagrange equations leads to a system of n coupled, second order nonlinear ordinary differential equations of the form

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} \quad = \quad \tau_i \quad i = 1, \ldots, n \tag{9.11}$$

The order, n, of the system is determined by the number of so-called **generalized coordinates** are required to describe the evolution of the system. We shall see that the n Denavit-Hartenberg joint variables serves as a set of generalized coordinates for an n-link rigid robot.

9.1.2 The General Case

Now, consider a system consisting of k particles, with corresponding position vectors $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_k$. If these particles are free to move about without any restrictions, then it is quite an easy

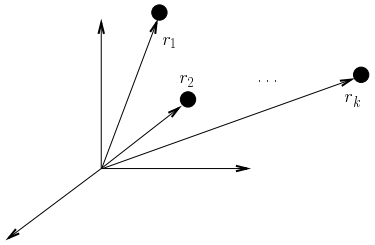

Figure 9.3: System of k particles

matter to describe their motion, by noting merely that the rate of change of the momentum of each mass equals the external force applied to it. However, if the motion of the particles is constrained in some fashion, then one must take into account not only the externally applied forces, but also the so-called constraint forces, that is, the forces needed to make the constraints hold. As a simple illustration of this, suppose the system consists of two particles, which are joined by a massless rigid wire of length ℓ. Then the two coordinates $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ must satisfy the constraint

$$\|\boldsymbol{r}_1 - \boldsymbol{r}_2\| = \ell, \quad \text{or} \quad (\boldsymbol{r}_1 - \boldsymbol{r}_2)^T(\boldsymbol{r}_1 - \boldsymbol{r}_2) = \ell^2. \tag{9.12}$$

If one applies some external forces to each particle, then the particles experience not only these external forces but also the force exerted by the wire, which is along the direction $\boldsymbol{r}_2 - \boldsymbol{r}_1$ and of appropriate magnitude. Therefore, in order to analyze the motion of the two particles, we can follow one of two options. First, we can compute, under each set of external forces, what the corresponding constraint force must be in order that the equation above continues to hold. Second, we can search for a method of analysis that does not require us to know the constraint force. Clearly, the second alternative is preferable, since it is in general quite an involved task to compute the constraint forces. The contents of this section are aimed at achieving this second objective.

First it is necessary to introduce some terminology. A constraint on the k coordinates $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_k$ is called **holonomic** if it is an equality constraint of the form

$$g_i(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_k) = 0, \qquad i = 1, \ldots, \ell \tag{9.13}$$

and **nonholonomic** otherwise. The constraint (9.12) imposed by connecting two particles by a massless rigid wire is a holonomic constraint. As as example of a nonholonomic constraint, consider a particle moving inside a sphere of radius p centered at the origin of the coordinate system. In this case the coordinate vector $\boldsymbol{r}$ of the particle must satisfy the constraint

$$\|\boldsymbol{r}\| \quad \leq \quad \rho. \tag{9.14}$$

Note that the motion of the particle is unconstrained so long as the particle remains away from the wall of the sphere; but when the particle comes into contact with the wall, it experiences a constraining force.

If a system is subjected to ℓ holonomic constraints, then one can think in terms of the constrained system having ℓ fewer degrees-of-freedom than the unconstrained system. In this case it may be possible to express the coordinates of the k particles in terms of n **generalized coordinates** $q_1, \ldots, q_n$. In other words, we assume that the coordinates of the various particles, subjected to the set of constraints (9.13), can be expressed in the form

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q_1, \ldots, q_n), \qquad i = 1, \ldots, k \tag{9.15}$$

where $q_1, \ldots, q_n$ are all independent. In fact, the idea of generalized coordinates can be used even when there are infinitely many particles. For example, a physical rigid object such as a bar contains an infinity of particles; but since the distance between each pair of particles is fixed throughout the motion of the bar, only six coordinates are sufficient to specify completely the coordinates of any particle in the bar. In particular, one could use three position coordinates to specify the location of the center of mass of the bar, and three Euler angles to specify the orientation of the body. To keep the discussion simple, however, we assume in what follows that the number of particles is finite. Typically, generalized coordinates are positions, angles, etc. In fact, in Chapter **??** we chose to denote the joint variables by the symbols $q_1, \ldots, q_n$ precisely because these joint variables form a set of generalized coordinates for an n-link robot manipulator.

One can now speak of **virtual displacements**, which are any set, $\delta \boldsymbol{r}_1, \ldots, \delta \boldsymbol{r}_k$, of infinitesimal displacements that are consistent with the constraints. For example, consider once again the constraint (9.12) and suppose $\boldsymbol{r}_1, \boldsymbol{r}_2$ are perturbed to $\boldsymbol{r}_1 + \delta \boldsymbol{r}_1$, $\boldsymbol{r}_2 + \delta \boldsymbol{r}_2$, respectively. Then, in order that the perturbed coordinates continue to satisfy the constraint, we must have

$$(\boldsymbol{r}_1 + \delta \boldsymbol{r}_1 - \boldsymbol{r}_2 - \delta \boldsymbol{r}_2)^T (\boldsymbol{r}_1 + \delta \boldsymbol{r}_1 - \boldsymbol{r}_2 - \delta \boldsymbol{r}_2) \quad = \quad \ell^2. \tag{9.16}$$

Now let us expand the above product and take advantage of the fact that the original coordinates $\boldsymbol{r}_1, \boldsymbol{r}_2$ satisfy the constraint (9.12); let us also neglect quadratic terms in $\delta \boldsymbol{r}_1, \delta \boldsymbol{r}_2$. This shows that

$$(\boldsymbol{r}_1 - \boldsymbol{r}_2)^T (\delta \boldsymbol{r}_1 - \delta \boldsymbol{r}_2) \quad = \quad 0. \tag{9.17}$$

Thus any perturbations in the positions of the two particles must satisfy the above equation in order that the perturbed positions continue to satisfy the constraint (9.12). Any pair of infinitesimal vectors $\delta \boldsymbol{r}_1, \delta \boldsymbol{r}_2$ that satisfy (9.17) would constitute a set of virtual displacements for this problem.

Now the reason for using generalized coordinates is to avoid dealing with complicated relationships such as (9.17) above. If (9.15) holds, then one can see that the set of all virtual displacements is precisely

$$\delta \boldsymbol{r}_i \quad = \quad \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_j} \delta q_j, \quad i = 1, \ldots, k \tag{9.18}$$

where the virtual displacements $\delta q_1, \ldots, \delta q_n$ of the generalized coordinates are unconstrained (that is what makes them generalized coordinates).

Next we begin a discussion of constrained systems in equilibrium. Suppose each particle is in equilibrium. Then the net force on each particle is zero, which in turn implies that the work done by each set of virtual displacements is zero. Hence the sum of the work done by any set of virtual displacements is also zero; that is,

$$\sum_{i=1}^{k} \boldsymbol{F}_i^T \delta \boldsymbol{r}_i \quad = \quad 0 \tag{9.19}$$

where $\boldsymbol{F}_i$ is the total force on particle i. As mentioned earlier, the force $\boldsymbol{F}_i$ is the sum of two quantities, namely (i) the externally applied force $\boldsymbol{f}_i$, and (ii) the constraint force $\boldsymbol{f}_i^{(a)}$. Now suppose that the total work done by the constraint forces corresponding to any set of virtual displacements is zero, that is,

$$\sum_{i=1}^{k} (\boldsymbol{f}_i^{(a)})^T \delta \boldsymbol{r}_i \quad = \quad 0. \tag{9.20}$$

This will be true whenever the constraint force between a pair of particles is directed along the radial vector connecting the two particles (see the discussion in the next paragraph). Substituting (9.20) into (9.19) results in

$$\sum_{i=1}^{k} \boldsymbol{f}_i^T \delta \boldsymbol{r}_i \;=\; 0. \tag{9.21}$$

The beauty of this equation is that it does not involve the unknown constraint forces, but only the known external forces. This equation expresses the **principle of virtual work**, which can be stated in words as follows: The work done by external forces corresponding to any set of virtual displacements is zero. Note that the principle is not universally applicable, but requires that (9.20) hold, that is, that the constraint forces do no work. Thus, if the principle of virtual work applies, then one can analyze the dynamics of a system *without* having to evaluate the constraint forces.

It is easy to verify that the principle of virtual work applies whenever the constraint force between a pair of particles acts along the vector connecting the position coordinates of the two particles. In particular, when the constraints are of the form (9.12), the principle applies. To see this, consider once again a single constraint of the form (9.12). In this case the constraint force, if any, must be exerted by the rigid massless wire, and therefore must be directed along the radial vector connecting the two particles. In other words, the force exerted on particle 1 by the wire must be of the form

$$\boldsymbol{f}_1^{(a)} \;=\; c(\boldsymbol{r}_1 - \boldsymbol{r}_2) \tag{9.22}$$

for some constant c (which could change as the particles move about). By the law of action and reaction, the force exerted on particle 2 by the wire must be just the negative of the above, that is,

$$\boldsymbol{f}_2^{(a)} \;=\; -c(\boldsymbol{r}_1 - \boldsymbol{r}_2). \tag{9.23}$$

Now the work done by the constraint forces corresponding to a set of virtual displacements is

$$(\boldsymbol{f}_1^{(a)})^T \delta \boldsymbol{r}_1 + (\boldsymbol{f}_2^{(a)})^T \delta \boldsymbol{r}_2 \;=\; c(\boldsymbol{r}_1 - \boldsymbol{r}_2)^T (\delta \boldsymbol{r}_1 - \delta \boldsymbol{r}_2). \tag{9.24}$$

But (9.17) shows that for any set of virtual displacements, the above inner product must be zero. Thus the principle of virtual work applies in a system constrained by (9.12). The same reasoning can be applied if the system consists of several particles, which are pairwise connected by rigid massless wires of fixed lengths, in which case the system is subjected to several constraints of the form (9.12). Now, the requirement that the motion of a body be rigid can be equivalently expressed as the requirement that the distance between any pair of points on the body remain constant as the body moves, that is, as an infinity of constraints of the form (9.12). Thus the principle of virtual work applies whenever rigidity is the only constraint on the motion. There are indeed situations when this principle does not apply,

typically in the presence of magnetic fields. However, in all situations encountered in this book, we can safely assume that the principle of virtual work is valid.

In (9.21), the virtual displacements $\delta \boldsymbol{r}_i$ are not independent, so we cannot conclude from this equation that each coefficient $\boldsymbol{F}_i$ *individually* equals zero. In order to apply such reasoning, we must transform to generalized coordinates. Before doing this, we consider systems that are not necessarily in equilibrium. For such systems, **D'Alembert's principle** states that, if one introduces a fictitious additional force $-\dot{\boldsymbol{p}}_i$ on particle i for each i, where $\boldsymbol{p}_i$ is the momentum of particle i, then each particle will be in equilibrium. Thus, if one modifies (9.19) by replacing $\boldsymbol{F}_i$ by $\boldsymbol{F}_i - \dot{\boldsymbol{p}}_i$, then the resulting equation is valid for arbitrary systems. One can then remove the constraint forces as before using the principle of virtual work. This results in the equations

$$\sum_{i=1}^{k} \boldsymbol{f}_i^T \delta \boldsymbol{r}_i - \sum_{i=1}^{k} \dot{\boldsymbol{p}}_i^T \delta \boldsymbol{r}_i \;=\; 0. \tag{9.25}$$

The above equation does not mean that each coefficient of $\delta \boldsymbol{r}_i$ is zero. For this purpose, express each $\delta \boldsymbol{r}_i$ in terms of the corresponding virtual displacements of generalized coordinates, as is done in (9.18). Then the virtual work done by the forces $\boldsymbol{f}_i$ is given by

$$\sum_{i=1}^{k} \boldsymbol{f}_i^T \delta \boldsymbol{r}_i \;=\; \sum_{i=1}^{k} \sum_{j=1}^{n} \boldsymbol{f}_i^T \frac{\partial \boldsymbol{r}_i}{\partial q_j} \delta q_j = \sum_{j=1}^{n} \psi_j \delta q_j \tag{9.26}$$

where

$$\psi_j \;=\; \sum_{i=1}^{k} \boldsymbol{f}_i^T \frac{\partial \boldsymbol{r}_i}{\partial q_j} \tag{9.27}$$

is called the j-th **generalized force**. Note that ψ_j need not have dimensions of force, just as q_j need not have dimensions of length; however, $\psi_j \delta q_j$ must always have dimensions of work.

Now let us study the second summation in (9.25) Since $\boldsymbol{p}_i = m_i \dot{\boldsymbol{r}}_i$, it follows that

$$\sum_{i=1}^{k} \dot{\boldsymbol{p}}_i^T \delta \boldsymbol{r}_i \;=\; \sum_{i=1}^{k} m_i \ddot{\boldsymbol{r}}_i^T \delta \boldsymbol{r}_i = \sum_{i=1}^{k} \sum_{j=1}^{n} m_i \ddot{\boldsymbol{r}}_i^T \frac{\partial \boldsymbol{r}_i}{\partial q_j} \delta q_j. \tag{9.28}$$

Next, using the product rule of differentiation, we see that

$$\sum_{i=1}^{k} m_i \ddot{\boldsymbol{r}}_i^T \frac{\partial \boldsymbol{r}_i}{\partial q_j} \;=\; \sum_{i=1}^{k} \left\{ \frac{d}{dt} \left[m_i \dot{\boldsymbol{r}}_i^T \frac{\partial \boldsymbol{r}_i}{\partial q_j} \right] - m_i \dot{\boldsymbol{r}}_i^T \frac{d}{dt} \left[\frac{\partial \boldsymbol{r}_i}{\partial q_j} \right] \right\}. \tag{9.29}$$

Now differentiate (9.15) using the chain rule; this gives

$$\boldsymbol{v}_i \;=\; \dot{\boldsymbol{r}}_i = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_j} \dot{q}_j. \tag{9.30}$$

Observe from the above equation that

$$\frac{\partial \boldsymbol{v}_i}{\partial \dot{q}_i} = \frac{\partial \boldsymbol{r}_i}{\partial q_j}. \tag{9.31}$$

Next,

$$\frac{d}{dt}\left[\frac{\partial \boldsymbol{r}_i}{\partial q_j}\right] = \sum_{\ell=1}^{n} \frac{\partial^2 \boldsymbol{r}_i}{\partial q_j \partial q_\ell}\dot{q}_\ell = \frac{\partial \boldsymbol{v}_i}{\partial q_j} \tag{9.32}$$

where the last equality follows from (9.30). Substituting from (9.31) and (9.32) into (9.29) and noting that $\dot{\boldsymbol{r}}_i = \boldsymbol{v}_i$ gives

$$\sum_{i=1}^{k} m_i \ddot{\boldsymbol{r}}_i^T \frac{\partial \boldsymbol{r}_i}{\partial q_j} = \sum_{i=1}^{k} \left\{ \frac{d}{dt}\left[m_i \boldsymbol{v}_i^T \frac{\partial \boldsymbol{v}_i}{\partial \dot{q}_j}\right] - m_i \boldsymbol{v}_i^T \frac{\partial \boldsymbol{v}_i}{\partial q_j}\right\}. \tag{9.33}$$

If we define the *kinetic energy* K to be the quantity

$$K = \sum_{i=1}^{k} \frac{1}{2} m_i \boldsymbol{v}_i^T \boldsymbol{v}_i \tag{9.34}$$

then the sum above can be compactly expressed as

$$\sum_{i=1}^{k} m_i \ddot{\boldsymbol{r}}_i^T \frac{\partial \boldsymbol{r}_i}{\partial q_j} = \frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j}. \tag{9.35}$$

Now, substituting from (9.35) into (9.28) shows that the second summation in (9.25) is

$$\sum_{i=1}^{k} \dot{\boldsymbol{p}}_i^T \delta \boldsymbol{r}_i = \sum_{j=1}^{n} \left\{ \frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j}\right\} \delta q_j. \tag{9.36}$$

Finally, combining (9.36) and (9.26) gives

$$\sum_{j=1}^{n} \left\{ \frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} - \psi_j \right\} \delta q_j = 0. \tag{9.37}$$

Now, since the virtual displacements δq_j are independent, we can conclude that each coefficient in (9.37) is zero, that is, that

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} = \psi_j, \quad j = 1, \ldots, n. \tag{9.38}$$

If the generalized force ψ_j is the sum of an externally applied generalized force and another one due to a potential field, then a further modification is possible. Suppose there exist functions τ_j and a potential energy function $P(q)$ such that

$$\psi_j = -\frac{\partial P}{\partial q_j} + \tau_j. \tag{9.39}$$

Then (9.38) can be written in the form

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = \tau_j \tag{9.40}$$

where $\mathcal{L} = K - P$ is the Lagrangian and we have recovered the **Euler-Lagrange equations of motion** as in Equation (9.11).

9.2 General Expressions for Kinetic and Potential Energy

In the previous section, we showed that the Euler-Lagrange equations can be used to derive the dynamical equations in a straightforward manner, provided one is able to express the kinetic and potential energy of the system in terms of a set of generalized coordinates. In order for this result to be useful in a practical context, it is therefore important that one be able to compute these terms readily for an n-link robotic manipulator. In this section we derive formulas for the kinetic energy and potential energy of a rigid robot using the Denavit-Hartenberg joint variables as generalized coordinates.

To begin we note that the kinetic energy of a rigid object is the sum of two terms: the translational energy obtained by concentrating the entire mass of the object at the center of mass, and the rotational kinetic energy of the body about the center of mass. Referring to Figure 9.4 we attach a coordinate frame at the center of mass (called the *body attached frame*) as shown. The kinetic energy of the rigid body is then given as

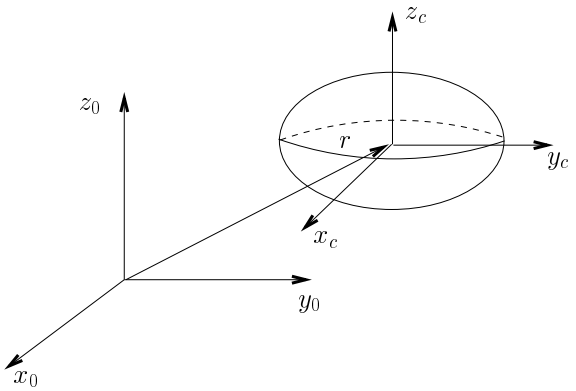

Figure 9.4: A General Rigid Body

$$\mathcal{K} = \frac{1}{2} m v^T v + \frac{1}{2} \boldsymbol{\omega}^T \mathcal{I} \boldsymbol{\omega}. \tag{9.41}$$

where m is the total mass of the object, v and ω are the linear and angular velocity vectors, respectively, and $\mathcal{I}$ is a symmetric 3×3 matrix called the **Inertia Tensor**.

9.2.1 The Inertia Tensor

It is understood that the linear and angular velocity vectors, v and ω, respectively, in the above expression for the kinetic energy are expressed in the inertial frame. In this case we know that ω is found from the skew symmetric matrix

$$S(\omega) = \dot{R}R^T \tag{9.42}$$

where R is the orientation transformation between the body attached frame and the inertial frame. It is therefore necessary to express the inertia tensor, $\mathcal{I}$, also in the inertial frame in order to compute the triple product $\omega^T \mathcal{I} \omega$. The inertia tensor relative to the inertial reference frame will depend on the configuration of the object. If we denote as I the inertia tensor expressed instead in the body attached frame, then the two matrices are related via a similarity transformation according to

$$\mathcal{I} = RIR^T \tag{9.43}$$

This is an important observation because the inertia matrix expressed in the body attached frame is a constant matrix independent of the motion of the object and easily computed. We next show how to compute this matrix explicitly.

Let the mass density of the object be represented as a function of position, $\rho(x, y, z)$. Then the inertia tensor in the body attached frame is computed as

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}. \tag{9.44}$$

where

$$\begin{aligned}
I_{xx} &= \int\int\int (y^2 + z^2)\rho(x, y, z)dx\ dy\ dz \\
I_{yy} &= \int\int\int (x^2 + z^2)\rho(x, y, z)dx\ dy\ dz \\
I_{zz} &= \int\int\int (x^2 + y^2)\rho(x, y, z)dx\ dy\ dz \\
I_{xy} = I_{yx} &= -\int\int\int xy\rho(x, y, z)dx\ dy\ dz \\
I_{xz} = I_{zx} &= -\int\int\int xz\rho(x, y, z)dx\ dy\ dz \\
I_{yz} = I_{zy} &= -\int\int\int yz\rho(x, y, z)dx\ dy\ dz
\end{aligned}$$

The integrals in the above expression are computed over the region of space occupied by the rigid body. The diagonal elements of the inertia tensor, I_{xx}, I_{yy}, I_{zz}, are called the **Principal Moments of Inertia** about the x,y,z axes, respectively. The off diagonal

terms I_{xy}, I_{xz}, etc., are called the **Cross Products of Inertia**. If the mass distribution of the body is symmetric with respect to the body attached frame then the cross products of inertia are identically zero.

Example: 9.2 *Uniform Rectangular Solid*

Consider the rectangular solid of length, a, width, b, and height, c, shown in Figure 9.5 and suppose that the density is constant, $\rho(x, y, z) = \rho$.

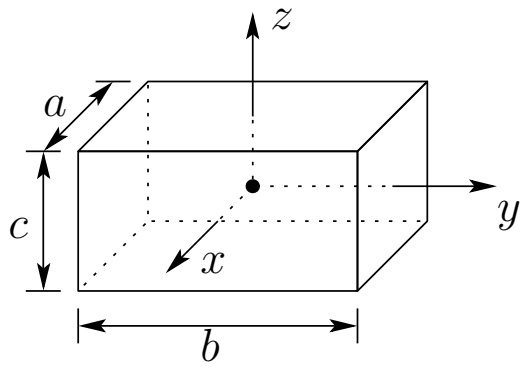

Figure 9.5: Uniform Rectangular Solid

If the body frame is attached at the geometric center of the object, then by symmetry, the cross products of inertia are all zero and it is a simple exercise to compute

$$I_{xx} = \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} (y^2 + z^2)\rho(x, y, z)dx\ dy\ dz = \rho \frac{abc}{12}(b^2 + c^2)$$

Likewise

$$I_{yy} = \rho \frac{abc}{12}(a^2 + c^2) \quad ; \quad I_{zz} = \rho \frac{abc}{12}(a^2 + b^2)$$

and the cross products of inertia are zero.

9.2.2 Kinetic Energy for an n-Link Robot

Now consider a manipulator consisting of n links. We have seen in Chapter 5 that the linear and angular velocities of any point on any link can be expressed in terms of the Jacobian matrix and the derivative of the joint variables. Since in our case the joint variables are indeed the generalized coordinates, it follows that, for appropriate Jacobian matrices J_{v_i} and J_{ω_i}, we have that

$$v_i = J_{v_i}(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \boldsymbol{\omega}_i = J_{\omega_i}(\boldsymbol{q})\dot{\boldsymbol{q}} \tag{9.45}$$

Now suppose the mass of link i is m_i and that the inertia matrix of link i, evaluated around a coordinate frame parallel to frame i but whose origin is at the center of mass, equals I_i. Then from (9.41) it follows that the overall kinetic energy of the manipulator equals

$$K = \frac{1}{2}\dot{\boldsymbol{q}}^T \sum_{i=1}^{n} \left[m_i J_{v_i}(\boldsymbol{q})^T J_{v_i}(\boldsymbol{q}) + J_{\omega_i}(\boldsymbol{q})^T R_i(\boldsymbol{q}) I_i R_i(\boldsymbol{q})^T J_{\omega_i}(\boldsymbol{q}) \right] \dot{\boldsymbol{q}} \tag{9.46}$$

In other words, the kinetic energy of the manipulator is of the form

$$K = \frac{1}{2}\dot{\boldsymbol{q}}^T D(\boldsymbol{q}) \dot{\boldsymbol{q}} \tag{9.47}$$

where $D(\boldsymbol{q})$ is a symmetric positive definite matrix that is in general configuration dependent. The matrix D is called the **inertia matrix**, and in Section 9.4 we will compute this matrix for several commonly occurring manipulator configurations.

9.2.3 Potential Energy for an n-Link Robot

Now consider the potential energy term. In the case of rigid dynamics, the only source of potential energy is gravity. The potential energy of the i-th link can be computed by assuming that the mass of the entire object is concentrated at its center of mass and is given by

$$P_i = g^T r_{ci} m_i. \tag{9.48}$$

where g is vector giving the direction of gravity in the inertial frame and the vector r_{ci} gives the coordinates of the center of mass of link i. The total potential energy of the n-link robot is therefore

$$P = \sum_{i=1}^{n} P_i = \sum_{i=1}^{n} g^T r_{ci} m_i. \tag{9.49}$$

In the case that the robot contains elasticity, for example, flexible joints, then the potential energy will include terms containing the energy stored in the elastic elements.

Note that the potential energy is a function only of the generalized coordinates and not their derivatives, i.e. the potential energy depends on the configuration of the robot but not on its velocity.

9.3 Equations of Motion

In this section, we specialize the Euler-Lagrange equations derived in Section 9.1 to the special case when two conditions hold: first, the kinetic energy is a quadratic function of the vector $\dot{\boldsymbol{q}}$ of the form

$$K = \frac{1}{2} \sum_{i,j}^{n} d_{ij}(\boldsymbol{q}) \dot{q}_i \dot{q}_j := \frac{1}{2}\dot{\boldsymbol{q}}^T D(\boldsymbol{q}) \dot{\boldsymbol{q}} \tag{9.50}$$

where the $n \times n$ "inertia matrix" $D(\boldsymbol{q})$ is symmetric and positive definite for each $q \in R^n$, and second, the potential energy $P = P(\boldsymbol{q})$ is independent of $\dot{q}$. We have already remarked that robotic manipulators satisfy this condition.

The Euler-Lagrange equations for such a system can be derived as follows. Since

$$L = K - P = \frac{1}{2}\sum_{i,j} d_{ij}(\boldsymbol{q})\dot{q}_i\dot{q}_j - P(\boldsymbol{q}) \tag{9.51}$$

we have that

$$\frac{\partial L}{\partial \dot{q}_k} = \sum_j d_{kj}\dot{q}_j \tag{9.52}$$

and

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} &= \sum_i d_{kj}\ddot{q}_j + \sum_j \frac{d}{dt}d_{kj}\dot{q}_j \\ &= \sum_j d_{kj}\ddot{q}_j + \sum_{i,j}\frac{\partial d_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j \end{aligned} \tag{9.53}$$

Also

$$\frac{\partial L}{\partial q_k} = \frac{1}{2}\sum_{i,j}\frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j - \frac{\partial P}{\partial q_k}. \tag{9.54}$$

Thus the Euler-Lagrange equations can be written

$$\sum_j d_{kj}\ddot{q}_j + \sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j - \frac{\partial P}{\partial q_k} = \tau_k. \tag{9.55}$$

By interchanging the order of summation and taking advantage of symmetry, we can show that

$$\sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i}\right\}\dot{q}_i\dot{q}_j = \frac{1}{2}\sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j}\right\}\dot{q}_i\dot{q}_j. \tag{9.56}$$

Hence

$$sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j = \sum_{i,j}\frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j. \tag{9.57}$$

The terms

$$c_{ijk} := \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\} \tag{9.58}$$

are known as **Christoffel symbols** (of the first kind). Note that, for a fixed k, we have $c_{ijk} = c_{jik}$, which reduces the effort involved in computing these symbols by a factor of about one half. Finally, if we define

$$\phi_k = \frac{\partial P}{\partial q_k} \tag{9.59}$$

then we can write the Euler-Lagrange equations as

$$\sum_i d_{kj}(\boldsymbol{q})\ddot{q}_j + \sum_{i,j} c_{ijk}(\boldsymbol{q})\dot{q}_i\dot{q}_j + \phi_k(\boldsymbol{q}) = \tau_k, \qquad k = 1, \ldots, n. \tag{9.60}$$

In the above equation, there are three types of terms. The first involve the second derivative of the generalized coordinates. The second are quadratic terms in the first derivatives of $\boldsymbol{q}$, where the coefficients may depend on $\boldsymbol{q}$. These are further classified into two types. Terms involving a product of the type $\dot{q}_i^2$ are called **centrifugal**, while those involving a product of the type $\dot{q}_i q_j$ where $i \neq j$ are called **Coriolis** terms. The third type of terms are those involving only $\boldsymbol{q}$ but not its derivatives. Clearly the latter arise from differentiating the potential energy. It is common to write (9.60) in matrix form as

$$D(\boldsymbol{q})\ddot{\boldsymbol{q}} + C(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} \tag{9.61}$$

where the k, j-th element of the matrix $C(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is defined as

$$c_{kj} = \sum_{i=1}^{n} c_{ijk}(\boldsymbol{q})\dot{q}_i \tag{9.62}$$

$$= \sum_{i=1}^{n} \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_j} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i. \tag{9.63}$$

Let us examine an important special case, where the *inertia matrix is diagonal and independent of* $\boldsymbol{q}$. In this case it follows from (9.58) that all of the Christoffel symbols are zero, since each d_{ij} is a constant. Moreover, the quantity d_{kj} is nonzero if and only if $k = j$, so that the Equations 9.60) decouple nicely into the form

$$d_{kk}\ddot{\boldsymbol{q}} - \phi_k(\boldsymbol{q}) = \tau_k, \qquad k = 1, \ldots, n. \tag{9.64}$$

In summary, the development in this section is very general and applies to *any* mechanical system whose kinetic energy is of the form (9.50) and whose potential energy is independent of $\dot{\boldsymbol{q}}$. In the next section we apply this discussion to study specific robot configurations.

9.4 Some Common Configurations

In this section we apply the above method of analysis to several manipulator configurations and derive the corresponding equations of motion. The configurations are progressively more complex, beginning with a two-link cartesian manipulator and ending with a five-bar linkage mechanism that has a particularly simple inertia matrix.

Two-Link Cartesian Manipulator

Consider the manipulator shown in Figure 9.6, consisting of two links and two prismatic

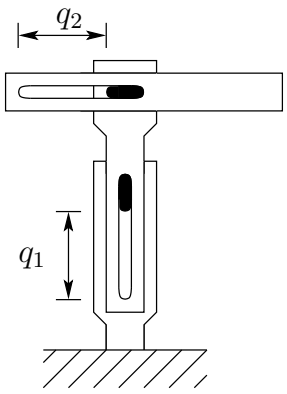

Figure 9.6: Two-link cartesian robot.

joints. Denote the masses of the two links by m_1 and m_2, respectively, and denote the displacement of the two prismatic joints by q_1 and q_2, respectively. Then it is easy to see, as mentioned in Section 9.1, that these two quantities serve as generalized coordinates for the manipulator. Since the generalized coordinates have dimensions of distance, the corresponding generalized forces have units of force. In fact, they are just the forces applied at each joint. Let us denote these by f_i, $i = 1, 2$.

Since we are using the joint variables as the generalized coordinates, we know that the kinetic energy is of the form (9.50) and that the potential energy is only a function of q_1 and q_2. Hence we can use the formulae in Section 9.3 to obtain the dynamical equations. Also, since both joints are prismatic, the angular velocity Jacobian is zero and the kinetic energy of each link consists solely of the translational term.

By (5.87) it follows that the velocity of the center of mass of link 1 is given by

$$\boldsymbol{v}_{c1} = J_{\boldsymbol{v}_{c1}} \dot{\boldsymbol{q}} \tag{9.65}$$

where

$$J_{\boldsymbol{v}_{c1}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \dot{\boldsymbol{q}} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}. \tag{9.66}$$

Similarly,

$$\boldsymbol{v}_{c2} = J_{\boldsymbol{v}_{c2}} \dot{\boldsymbol{q}} \tag{9.67}$$

where

$$J_{\boldsymbol{v}_{c2}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{9.68}$$

Hence the kinetic energy is given by

$$K = \frac{1}{2}\dot{\boldsymbol{q}}^T \left\{ m_1 J_{\boldsymbol{v}_c}^T J_{\boldsymbol{v}_{c1}} + m_2 J_{\boldsymbol{v}_{c2}}^T J_{\boldsymbol{v}_{c2}} \right\} \dot{\boldsymbol{q}}. \tag{9.69}$$

Comparing with (9.50), we see that the inertia matrix D is given simply by

$$D = \begin{bmatrix} m_1 + m_2 & 0 \\ 0 & m_2 \end{bmatrix}. \tag{9.70}$$

Next, the potential energy of link 1 is $m_1 g q_1$, while that of link 2 is $m_2 g q_1$, where g is the acceleration due to gravity. Hence the overall potential energy is

$$P = g(m_1 + m_2)q_1. \tag{9.71}$$

Now we are ready to write down the equations of motion. Since the inertia matrix is constant, all Christoffel symbols are zero. Further, the vectors ϕ_k are given by

$$\phi_1 = \frac{\partial P}{\partial q_1} = g(m_1 + m_2), \qquad \phi_2 = \frac{\partial P}{\partial q_2} = 0. \tag{9.72}$$

Substituting into (9.60) gives the dynamical equations as

$$\begin{aligned} (m_1 + m_2)\ddot{q}_1 + g(m_1 + m_2) &= f_1 \\ m_2 \ddot{q}_2 &= f_2. \end{aligned} \tag{9.73}$$

Planar Elbow Manipulator

Now consider the planar manipulator with two revolute joints shown in Figure 9.7. Let

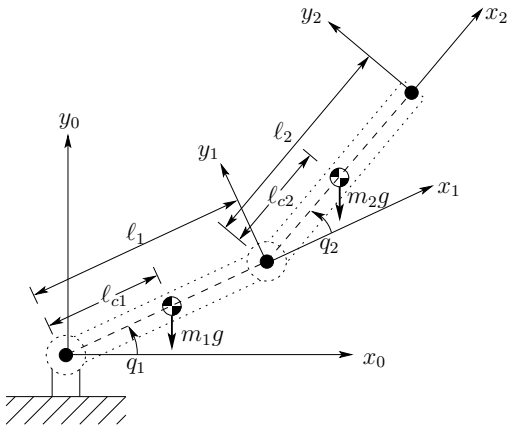

Figure 9.7: Two-link revolute joint arm.

us fix notation as follows: For $i = 1, 2$, q_i denotes the joint angle, which also serves as a

generalized coordinate; m_i denotes the mass of link i, ℓ_i denotes the length of link i; ℓ_{ci} denotes the distance from the previous joint to the center of mass of link i; and I_i denotes the moment of inertia of link i about an axis coming out of the page, passing through the center of mass of link i.

We will make effective use of the Jacobian expressions in Chapter 5 in computing the kinetic energy. Since we are using joint variables as the generalized coordinates, it follows that we can use the contents of Section 9.7. First,

$$\boldsymbol{v}_{c1} = J_{\boldsymbol{v}_{c1}} \dot{\boldsymbol{q}} \tag{9.74}$$

where, from (5.80),

$$J_{\boldsymbol{v}_{c1}} = \begin{bmatrix} -\ell_c \sin q_1 & 0 \\ \ell_{c1} \cos q_1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{9.75}$$

Similarly,

$$\boldsymbol{v}_{c2} = J_{\boldsymbol{v}_{c2}} \dot{\boldsymbol{q}} \tag{9.76}$$

where

$$J_{\boldsymbol{v}_{c2}} = \begin{bmatrix} -\ell_1 \sin q_1 - \ell_{c2} \sin(q_1 + q_2) & -\ell_{c2} \sin(q_1 + q_2) \\ \ell_1 \cos q_1 + \ell_{c2} \cos(q_1 + q_2) & \ell_{c2} \cos(q_1 + q_2) \\ 0 & 0 \end{bmatrix}. \tag{9.77}$$

Hence the translational part of the kinetic energy is

$$\frac{1}{2} m_1 \boldsymbol{v}_{c1}^T \boldsymbol{v}_{c1} + \frac{1}{2} m_2 \boldsymbol{v}_{c2}^T \boldsymbol{v}_{c2} = \frac{1}{2} \dot{\boldsymbol{q}} \left\{ m_1 J_{\boldsymbol{v}_{c1}}^T J_{\boldsymbol{v}_{c1}} + m_2 J_{\boldsymbol{v}_{c2}}^T J_{\boldsymbol{v}_{c2}} \right\} \dot{\boldsymbol{q}}. \tag{9.78}$$

Next we deal with the angular velocity terms. Because of the particularly simple nature of this manipulator, many of the potential difficulties do not arise. First, it is clear that

$$\boldsymbol{\omega}_1 = \dot{q}_1 \boldsymbol{k}, \qquad \boldsymbol{\omega}_2 = (\dot{q}_1 + \dot{q}_2) \boldsymbol{k} \tag{9.79}$$

when expressed in the base inertial frame. Moreover, since $\boldsymbol{\omega}_i$ is aligned with $\boldsymbol{k}$, the triple product $\boldsymbol{\omega}_i^T I_i \boldsymbol{\omega}_i$ reduces simply to $(I_{33})_i$ times the square of the magnitude of the angular velocity. This quantity $(I_{33})_i$ is indeed what we have labeled as I_i above. Hence the rotational kinetic energy of the overall system is

$$\frac{1}{2} \dot{\boldsymbol{q}}^T \left\{ I_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + I_2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\} \dot{\boldsymbol{q}} \tag{9.80}$$

Now we are ready to form the inertia matrix $D(\boldsymbol{q})$. For this purpose, we merely have to add the two matrices in (9.78) and (9.80), respectively. Thus

$$D(\boldsymbol{q}) = m_1 J_{\boldsymbol{v}_{c1}}^T J_{\boldsymbol{v}_{c1}} + m_2 J_{\boldsymbol{v}_{c2}}^T J_{\boldsymbol{v}_{c2}} + \begin{bmatrix} I_1 + I_2 & I_2 \\ I_2 & I_2 \end{bmatrix}. \tag{9.81}$$

Carrying out the above multiplications and using the standard trigonometric identities $\cos^2\theta + \sin^2\theta = 1$, $\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$ leads to

$$\begin{aligned} d_{11} &= m_1\ell_{c1}^2 + m_2(\ell_1^2 + \ell_{c2}^2 + 2\ell_1\ell_{c2}^2 + 2\ell_1\ell_{c2}\cos q_2) + I_1 + I_2 \\ d_{12} &= d_{21} = m_2(\ell_{c2}^2 + \ell_1\ell_{c2}\cos q_2) + I_2 \\ d_{22} &= m_2\ell_{c2}^2 + I_2. \end{aligned} \tag{9.82}$$

Now we can compute the Christoffel symbols using the definition (9.58). This gives

$$\begin{aligned} c_{111} &= \frac{1}{2}\frac{\partial d_{11}}{\partial q_1} = 0 \\ c_{121} &= c_{211} = \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = -m_2\ell_1\ell_{c2}\sin q_2 =: h \\ c_{221} &= \frac{\partial d_{12}}{\partial q_2} - \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = h \\ c_{112} &= \frac{\partial d_{21}}{\partial q_1} - \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = -h \\ c_{122} &= c_{212} = \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = 0 \\ c_{222} &= \frac{1}{2}\frac{\partial d_{22}}{\partial q_2} = 0. \end{aligned} \tag{9.83}$$

Next, the potential energy of the manipulator is just the sum of those of the two links. For each link, the potential energy is just its mass multiplied by the gravitational acceleration and the height of its center of mass. Thus

$$\begin{aligned} P_1 &= m_1 g\ell_{c1}\sin q_1 \\ P_2 &= m_2 g(\ell_1\sin q_1 + \ell_{c2}\sin(q_1 + q_2)) \\ P &= P_1 + P_2 = (m_1\ell_{c1} + m_2\ell_1)g\sin q_1 + m_2\ell_{c2}g\sin(q_1 + q_2). \end{aligned} \tag{9.84}$$

Hence, the functions ϕ_k defined in (9.59) become

$$\phi_1 = \frac{\partial P}{\partial q_1} = (m_1\ell_{c1} + m_2\ell_1)g\cos q_1 + m_2\ell_{c2}g\cos(q_1 + q_2) \tag{9.85}$$

$$\phi_2 = \frac{\partial P}{\partial q_2} = m_2\ell_{c2}\cos(q_1 + q_2). \tag{9.86}$$

Finally we can write down the dynamical equations of the system as in (9.60). Substituting for the various quantities in this equation and omitting zero terms leads to

$$\begin{aligned} d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + c_{121}\dot{q}_1\dot{q}_2 + c_{211}\dot{q}_2\dot{q}_1 + c_{221}\dot{q}_2^2 + \phi_1 &= \tau_1 \\ d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 + c_{112}\dot{q}_1^2 + \phi_2 &= \tau_2. \end{aligned} \tag{9.87}$$

In this case the matrix $C(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is given as

$$C = \begin{bmatrix} h\dot{q}_2 & h\dot{q}_2 + h\dot{q}_1 \\ -h\dot{q}_1 & 0 \end{bmatrix}. \tag{9.88}$$

Planar Elbow Manipulator with Remotely Driven Link

Now we illustrate the use of Lagrangian equations in a situation where the generalized coordinates are not the joint variables defined in earlier chapters. Consider again the planar elbow manipulator, but suppose now that both joints are driven by motors mounted at the base. The first joint is turned directly by one of the motors, while the other is turned via a gearing mechanism or a timing belt (see Figure 9.8). In this case one should choose

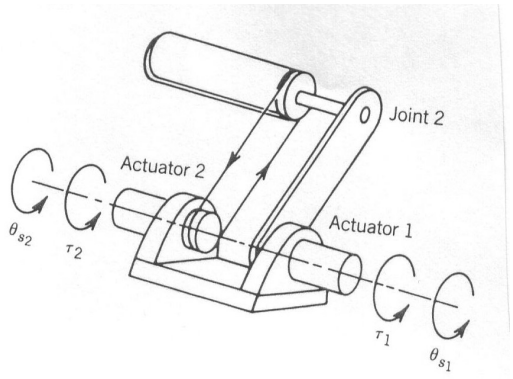

Figure 9.8: Two-link revolute joint arm with remotely driven link.

the generalized coordinates as shown in Figure 9.9, because the angle p_2 is determined by

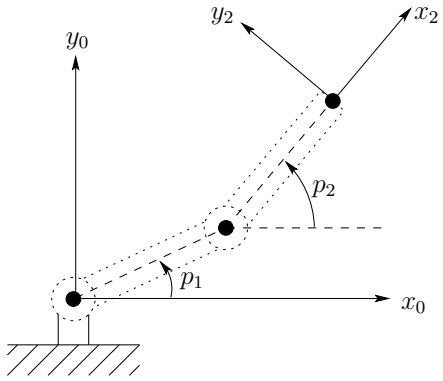

Figure 9.9: Generalized coordinates for robot of Figure 6.4.

driving motor number 2, and is not affected by the angle p_1. We will derive the dynamical equations for this configuration, and show that some simplifications will result.

Since p_1 and p_2 are not the joint angles used earlier, we cannot use the velocity Jacobians derived in Chapter 5 in order to find the kinetic energy of each link. Instead, we have to

carry out the analysis directly. It is easy to see that

$$\boldsymbol{v}_{c1} = \begin{bmatrix} -\ell_{c1}\sin p_1 \\ \ell_{c1}\cos p_1 \\ 0 \end{bmatrix} \dot{p}_1, \quad \boldsymbol{v}_{c2} = \begin{bmatrix} \ell_1 \sin p_1 & -\ell_{c2}\sin p_2 \\ \ell_1 \cos p_1 & \ell_{c2}\cos p_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} \tag{9.89}$$

$$\boldsymbol{\omega}_1 = \dot{p}_1 \boldsymbol{k}, \qquad \boldsymbol{\omega}_2 = \dot{p}_2 \boldsymbol{k}. \tag{9.90}$$

Hence the kinetic energy of the manipulator equals

$$K = \frac{1}{2}\dot{\boldsymbol{p}}^T D(\boldsymbol{p})\dot{\boldsymbol{p}} \tag{9.91}$$

where

$$D(\boldsymbol{p}) = \begin{bmatrix} m_1\ell_{c1}^2 + m_2\ell_1^2 + I_1 & m_2\ell_1\ell_{c2}\cos(p_2 - p_1) \\ m_2\ell_1\ell_{c2}\cos(p_2 - p_1) & m_2\ell_{c2}^2 + I_2. \end{bmatrix} \tag{9.92}$$

Computing the Christoffel symbols as in (9.58) gives

$$\begin{aligned} c_{111} &= \frac{1}{2}\frac{\partial d_{11}}{\partial p_1} = 0 \\ c_{121} &= c_{211} = \frac{1}{2}\frac{\partial d_{11}}{\partial p_2} = 0 \\ c_{221} &= \frac{\partial d_{12}}{\partial p_2} - \frac{1}{2}\frac{\partial d_{22}}{\partial p_1} = -m_2\ell_1\ell_{c2}\sin(p_2 - p_1) \\ c_{112} &= \frac{\partial d_{21}}{\partial p_1} - \frac{1}{2}\frac{\partial d_{11}}{\partial p_2} = m_2\ell_1\ell_{c2}\sin(p_2 - p_1) \\ c_{212} &= = c_{122} = \frac{1}{2}\frac{\partial d_{22}}{\partial p_1} = 0 \\ c_{222} &= \frac{1}{2}\frac{\partial d_{22}}{\partial p_2} = 0. \end{aligned} \tag{9.93}$$

Next, the potential energy of the manipulator, in terms of p_1 and p_2, equals

$$P = m_1 g \ell_{c1}\sin p_1 + m_2 g(\ell_1 \sin p_1 + \ell_{c2}\sin p_2). \tag{9.94}$$

Hence the gravitational generalized forces are

$$\begin{aligned} \phi_1 &= (m_1\ell_{c1} + m_2\ell_1)g\cos p_1 \\ \phi_2 &= m_2\ell_{c2}g\cos p_2. \end{aligned}$$

Finally, the equations of motion are

$$\begin{aligned} d_{11}\ddot{p}_1 + d_{12}\ddot{p}_2 + c_{221}\dot{p}_2^2 + \phi_1 &= \tau_1 \\ d_{21}\ddot{p}_1 + d_{22}\ddot{p}_2 + c_{112}\dot{p}_1^2 + \phi_2 &= \tau_2. \end{aligned} \tag{9.95}$$

Comparing (9.95) and (9.87), we see that by driving the second joint remotely from the base we have eliminated the Coriolis forces, but we still have the centrifugal forces coupling the two joints.

Five-Bar Linkage

Now consider the manipulator shown in Figure 9.10. We will show that, if the parameters

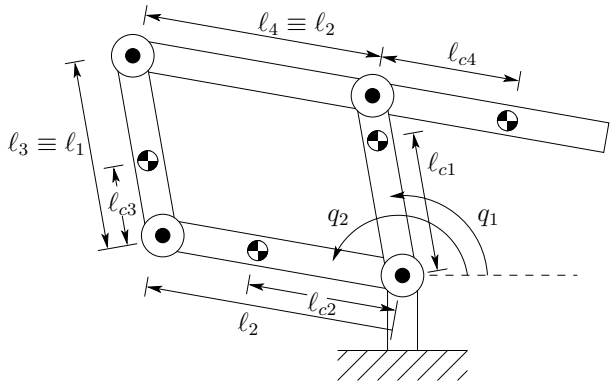

Figure 9.10: Five-bar linkage.

of the manipulator satisfy a simple relationship, then the equations of the manipulator are decoupled, so that each quantity q_1 and q_2 can be controlled independently of the other. The mechanism in Figure 9.10 is called a **five-bar linkage**. Clearly there are only four bars in the figure, but in the theory of mechanisms it is a convention to count the ground as an additional linkage, which explains the terminology. In Figure 9.10, it is assumed that the lengths of links and 3 are the same, and that the two lengths marked ℓ_2 are the same; in this way the closed path in the figure is in fact a parallelogram, which greatly simplifies the computations. Notice, however, that the quantities ℓ_{c1}, and ℓ_{c3} need not be equal. For example, even though links 1 and 3 have the same length, they need not have the same mass distribution.

It is clear from the figure that, even though there are four links being moved, there are in fact only two degrees-of-freedom, identified as q_1 and q_2. Thus, in contrast to the earlier mechanisms studied in this book, this one is a closed kinematic chain (though of a particularly simple kind). As a result, we cannot use the earlier results on Jacobian matrices, and instead have to start from scratch. As a first step we write down the coordinates of the

centers of mass of the various links as a function of the generalized coordinates. This gives

$$\begin{bmatrix} x_{c1} \\ y_{c1} \end{bmatrix} = \begin{bmatrix} \ell_{c1}\cos q_1 \\ \ell_{c1}\sin q_1 \end{bmatrix} \tag{9.96}$$

$$\begin{bmatrix} x_{c2} \\ y_{c2} \end{bmatrix} = \begin{bmatrix} \ell_{c2}\cos q_2 \\ \ell_{c2}\sin q_2 \end{bmatrix} \tag{9.97}$$

$$\begin{bmatrix} x_{c3} \\ y_{c3} \end{bmatrix} = \begin{bmatrix} \ell_{c2}\cos q_1 \\ \ell_{c2}\sin q_2 \end{bmatrix} + \begin{bmatrix} \ell_{c3}\cos q_1 \\ \ell_{c3}\sin q_1 \end{bmatrix} \tag{9.98}$$

$$\begin{aligned} \begin{bmatrix} x_{c4} \\ y_{c4} \end{bmatrix} &= \begin{bmatrix} \ell_1\cos q_1 \\ \ell_1\sin q_1 \end{bmatrix} + \begin{bmatrix} \ell_{c4}\cos(q_2-\pi) \\ \ell_{c4}\sin(q_2-\pi) \end{bmatrix} \\ &= \begin{bmatrix} \ell_1\cos q_1 \\ \ell_1\sin q_1 \end{bmatrix} - \begin{bmatrix} \ell_{c4}\cos q_2 \\ \ell_{c4}\sin q_2 \end{bmatrix}. \end{aligned} \tag{9.99}$$

Next, with the aid of these expressions, we can write down the velocities of the various centers of mass as a function of $\dot{q}_1$ and $\dot{q}_2$. For convenience we drop the third row of each of the following Jacobian matrices as it is always zero. The result is

$$\begin{aligned} \boldsymbol{v}_{c1} &= \begin{bmatrix} -\ell_{c1}\sin q_1 & 0 \\ \ell_{c1}\cos q_1 & 0 \end{bmatrix} \dot{\boldsymbol{q}} \\ \boldsymbol{v}_{c2} &= \begin{bmatrix} 0 & -\ell_{c2}\sin q_2 \\ 0 & \ell_{c2}\cos q_2 \end{bmatrix} \dot{\boldsymbol{q}} \\ \boldsymbol{v}_{c3} &= \begin{bmatrix} -\ell_{c3}\sin q_1 & -\ell_2\sin q_2 \\ \ell_{c3}\cos q_1 & \ell_2\cos q_2 \end{bmatrix} \dot{\boldsymbol{q}} \\ \boldsymbol{v}_{c4} &= \begin{bmatrix} -\ell_1\sin q_1 & \ell_{c4}\sin q_2 \\ \ell_1\cos q_1 & \ell_{c4}\cos q_2 \end{bmatrix} \dot{\boldsymbol{q}}. \end{aligned} \tag{9.100}$$

(9.101)

Let us define the velocity Jacobians $J_{\boldsymbol{v}_{ci}}$, $i = 1, \ldots, 4$ in the obvious fashion, that is, as the four matrices appearing in the above equations. Next, it is clear that the angular velocities of the four links are simply given by

$$\boldsymbol{\omega}_1 = \boldsymbol{\omega}_3 = \boldsymbol{q}_1\boldsymbol{k}, \boldsymbol{\omega}_2 = \boldsymbol{\omega}_4 = \dot{q}_2\boldsymbol{k}. \tag{9.102}$$

Thus the inertia matrix is given by

$$D(\boldsymbol{q}) = \sum_{i=1}^{4} m_i J_{vc}^T J_{vc} + \begin{bmatrix} I_1 + I_3 & 0 \\ 0 & I_2 + I_4 \end{bmatrix}. \tag{9.103}$$

If we now substitute from (9.100) into the above equation and use the standard trigonometric identities, when the dust settles we are left with

$$\begin{aligned} d_{11}(\boldsymbol{q}) &= m_1\ell_{c1}^2 + m_3\ell_{c3}^2 + m_4\ell_1^2 + I_1 + I_3 \\ d_{12}(\boldsymbol{q}) &= d_{21}(\boldsymbol{q}) = (m_3\ell_2\ell_{c3} - m_4\ell_1\ell_{c4})\cos(q_2 - q_1) \\ d_{22}(\boldsymbol{q}) &= m_2\ell_{c2}^2 + m_3\ell_2^2 + m_4\ell_{c4}^2 + I_2 + I_4. \end{aligned} \tag{9.104}$$

Now the thing to note is that if

$$m_3\ell_2\ell_{c3} = m_4\ell_1\ell_{c4} \tag{9.105}$$

then the inertia matrix is diagonal and constant, and as a consequence the dynamical equations will contain neither Coriolis nor centrifugal terms.

Turning now to the potential energy, we have that

$$\begin{aligned} P &= g\sum_{i=1}^{4} \boldsymbol{y}_{ci} \\ &= g\sin q_1(m_1\ell_{c1} + m_3\ell_{c3} + m_4\ell_1) \\ &= g\sin q_2(m_2\ell_{c2} + m_3\ell_2 - m_4\ell_{c4}). \end{aligned} \tag{9.106}$$

Hence

$$\begin{aligned} \phi_1 &= g\cos q_1(m_1\ell_{c1} + m_3\ell_{c3} + m_4\ell_1) \\ \phi_2 &= g\cos q_2(m_2\ell_{c2} + m_3\ell_2 - m_4\ell_{c4}). \end{aligned} \tag{9.107}$$

Notice that ϕ_1 depends only on q_1 but not on q_2 and similarly that ϕ_2 depends only on q_2 but not on q_1. Hence, if the relationship (9.105) is satisfied, then the rather complex-looking manipulator in Figure 9.10 is described by the **decoupled** set of equations

$$d_{11}\ddot{q}_1 + \phi_1(q_1) = \tau_1, \qquad d_{22}\ddot{q}_2 + \phi_2(q_2) = \tau_2. \tag{9.108}$$

This discussion helps to explain the increasing popularity of the parallelogram configuration in industrial robots (e.g., P-50). If the relationship (9.105) is satisfied, then one can adjust the two angles q_1 and q_2 independently, without worrying about interactions between the two angles. Compare this with the situation in the case of the planar elbow manipulators discussed earlier in this section.

9.5 Properties of Robot Dynamic Equations

The equations of motion for an n-link robot can be quite formidable especially if the robot contains one or more revolute joints. Fortunately, these equations contain some important structural properties which can be exploited to good advantage in particular for developing control algorithms. We will see this in subsequent chapters. Here we will discuss some of these properties, the most important of which are the so-called **skew symmetry** property and the related **passivity property**, and the **linearity in the parameters** property. For revolute joint robots, the inertia matrix also satisfies global bounds that are useful for control design.

9.5.1 The Skew Symmetry and Passivity Properties

The **Skew Symmetry** property refers to an important relationship between the inertia matrix $D(\boldsymbol{q})$ and the matrix $C(\boldsymbol{q}, \dot{\boldsymbol{q}})$ appearing in (9.61) that will be of fundamental importance for the problem of manipulator control considered in later chapters.

Proposition: 9.1 *Define the matrix $N(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{D}(\boldsymbol{q}) - 2C(\boldsymbol{q}, \dot{\boldsymbol{q}})$. Then $N(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is skew symmetric, that is, the components n_{jk} of N satisfy $n_{jk} = -n_{kj}$.*

Proof: Given the inertia matrix $D(\boldsymbol{q})$, the kj-th component of $\dot{D}(\boldsymbol{q})$ is given by the chain rule as

$$\dot{d}_{kj} = \sum_{i=1}^{n} \frac{\partial d_{kj}}{\partial q_i} \dot{q}_i. \tag{9.109}$$

Therefore, the kj-th component of $N = \dot{D} - 2C$ is given by

$$\begin{aligned} n_{kj} &= \dot{d}_{kj} - 2c_{kj} \\ &= \sum_{i=1}^{n} \left[\frac{\partial d_{kj}}{\partial q_i} - \left\{ \frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial_j} - \frac{\partial d_{ij}}{\partial q_k} \right\} \right] \dot{q}_i \\ &= \sum_{i=1}^{n} \left[\frac{\partial d_{ij}}{\partial q_k} - \frac{\partial d_{ki}}{\partial q_j} \right] \dot{q}_i. \end{aligned} \tag{9.110}$$

Since the inertia matrix $D(\boldsymbol{q})$ is symmetric, that is, $d_{ij} = d_{ji}$, it follows from (9.110) by interchanging the indices k and j that

$$n_{jk} = -n_{kj} \tag{9.111}$$

which completes the proof.

Related to the skew symmetry property is the so-called **Passivity Property** which, in the present context, means that there exists a constant, $\beta \geq 0$, such that

$$\int_0^T \dot{q}^T(\zeta)\tau(\zeta)d\zeta \geq -\beta, \quad \forall\ T > 0. \tag{9.112}$$

The term $\dot{q}^T\tau$ has units of power hence, the expression $\int_0^T \dot{q}^T(\zeta)\tau(\zeta)d\zeta$ is the energy produced by the system over the time interval $[0, T]$. Passivity therefore means that the amount of energy produced by the system has a lower bound given by $-\beta$. The word passivity comes from circuit theory where a passive system according to the above definition is one that can be built from passive components (resistors, capacitors, inductors). Likewise a passive mechanical system can be built from masses, springs, and dampers.

To prove the passivity property, let H be the total energy of the system, i.e., the sum of the kinetic and potential energies,

$$H = \frac{1}{2}\dot{q}^T D(q)\dot{q} + P(q) \tag{9.113}$$

Then, the derivative $\dot{H}$ satisfies

$$\dot{H} = \dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{D}(q)\dot{q} + \dot{q}^T \frac{\partial P}{\partial q} \tag{9.114}$$

$$= \dot{q}^T\{\tau - C(q,\dot{q}) - g(q)\} + \frac{1}{2}\dot{q}^T \dot{D}(q)\dot{q} + \dot{q}^T \frac{\partial P}{\partial q} \tag{9.115}$$

where we have substituted for $D(q)\ddot{q}$ using the equations of motion. Collecting terms and using the fact that $g(q) = \frac{\partial P}{\partial q}$ yields

$$\dot{H} = \dot{q}^T\tau + \frac{1}{2}\dot{q}^T\{\dot{D}(q) - 2C(q,\dot{q})\}\dot{q} \tag{9.116}$$

$$= \dot{q}^T\tau \tag{9.117}$$

the latter equality following from the skew-symmetry property. Integrating both sides of (9.117) with respect to time gives,

$$\int_0^T \dot{q}^T(\zeta)\tau(\zeta)d\zeta = H(T) - H(0) \geq -H(0) \tag{9.118}$$

since the total energy $H(T)$ is non–negative, and the passivity property therefore follows with $\beta = H(0)$.

9.5.2 Bounds on the Inertia Matrix

We have remarked previously that the inertia matrix for an n-link rigid robot is symmetric and positive definite. For a fixed value of the generalized coordinate q, let $0 < \lambda_1(q) \leq \ldots, \leq \lambda_n(q)$ denote the n eigenvalues of $D(q)$. These eigenvalues are positive as a consequence of the positive definiteness of $D(q)$. As a result, it can easily be shown that

$$\lambda_1(q)I_{n\times n} \leq D(q) \leq \lambda_n(q)I_{n\times n} \tag{9.119}$$

where $I_{n\times n}$ denotes the $n \times n$ identity matrix. The above inequalities are interpreted in the standard sense of matrix inequalities, namely, if A and B are $n \times n$ matrices, then $B < A$ means that the matrix $A - B$ is positive definite and $B \leq A$ means that $A - B$ is positive semi-definite.

If all of the joints are revolute then the inertia matrix contains only bounded functions of the joint variables, i.e., terms containing sine and cosine functions. As a result one can find constants λ_m and λ_M that provide uniform (independent of q) bounds in the inertia matrix

$$\lambda_m I_{n\times n} \leq D(q) \leq \lambda_M I_{n\times n} < \infty \tag{9.120}$$

9.5.3 Linearity in the Parameters

The robot equations of motion are defined in terms of certain parameters, such as link masses, moments of inertia, etc., that must be determined for each particular robot in order, for example, to simulate the equations or to tune controllers. The complexity of the dynamic equations makes the determination of these parameters a difficult task. Fortunately, the equations of motion are linear in these inertia parameters in the following sense. There exists an $n \times \ell$ function, $Y(q, \dot{q}, \ddot{q})$, which we assume is completely known, and an ℓ-dimensional vector Θ such that the Euler-Lagrange equations can be written

$$D(q) + C(q, \dot{q})\dot{q} + g(q) =: Y(q, \dot{q}, \ddot{q})\Theta = \tau \tag{9.121}$$

The function, $Y(q, \dot{q}, \ddot{q})$, is called the **Regressor** and Θ is the **Parameter** vector. The dimension of the parameter space, $\mathbf{R}^\ell$, i.e., the number of parameters needed to write the dynamics in this way, is not unique. In general, a given rigid body is described by ten parameters, namely, the total mass, the six independent entries of the inertia tensor, and the three coordinates of the center of mass. An n-link robot then has a maximum of $10n$ independent dynamics parameters. However, since the link motions are constrained and coupled by the joint interconnections, there are actually fewer than $10n$ independent parameters. Finding a minimal set of parameters that can parametrize the dynamic equations is, however, difficult in general.

Example: 9.3 *Two Link Planar Robot*

Consider the two link, revolute joint, planar robot from section 9.4 above. If we group the inertia terms appearing in Equation 9.82 as

$$\Theta_1 = m_1\ell_{c1}^2 + m_2(\ell_1^2 + \ell_{c2}^2) + I_1 + I_2 \tag{9.122}$$
$$\Theta_2 = m_2\ell_1\ell_{c2} \tag{9.123}$$
$$\Theta_3 = m_2\ell_1\ell_{c2} \tag{9.124}$$

then we can write the inertia matrix elements as

$$d_{11} = \Theta_1 + 2\Theta_2\cos(q_2) \tag{9.125}$$
$$d_{12} = d_{21} = \Theta_3 + \Theta_2\cos(q_2) \tag{9.126}$$
$$d_{22} = \Theta_3 \tag{9.127}$$

No additional parameters are required in the Christoffel symbols as these are functions of the elements of the inertia matrix. The gravitational torques require additional parameters, in general. Setting

$$\Theta_4 = m_1\ell_{c1} + m_2\ell_1 \tag{9.128}$$
$$\Theta_5 = m_2\ell_2 \tag{9.129}$$

we can write the gravitational terms, ϕ_1 and ϕ_2 as

$$\phi_1 = \Theta_4 g\cos(q_1) + \Theta_5 g\cos(q_1+q_2) \tag{9.130}$$
$$\phi_2 = \Theta_5 \cos(q_1+q_2) \tag{9.131}$$

Substituting these into the equations of motion it is straightforward to write the dynamics in the form (9.121) where

$$Y(q,\dot{q},\ddot{q}) = \tag{9.132}$$
$$\begin{bmatrix} \ddot{q}_1 & \cos(q_2)(2\ddot{q}_1+\ddot{q}_2)+\sin(q_2)(\dot{q}_1^2-2\dot{q}_1\dot{q}_2) & \ddot{q}_2 & g\cos(q_1) & g\cos(q_1+q_2) \\ 0 & \cos(q_2)\ddot{q}_1+\sin(q_2)\dot{q}_1^2 & \ddot{q}_2 & 0 & g\cos(q_1+q_2) \end{bmatrix}$$

and the parameter vector Θ is given by

$$\Theta = \begin{bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \\ \Theta_4 \\ \Theta_5 \end{bmatrix} = \begin{bmatrix} m_1\ell_{c1}^2 + m_2(\ell_1^2+\ell_{c2}^2) + I_1 + I_2 \\ m_2\ell_1\ell_{c2} \\ m_2\ell_1\ell_{c2} \\ m_1\ell_{c1} + m_2\ell_1 \\ m_2\ell_2 \end{bmatrix} \tag{9.133}$$

Thus, we have parameterized the dynamics using a five dimensional parameter space. Note that in the absence of gravity, as in a SCARA configuration, only three parameters are needed.

9.6 Newton-Euler Formulation

In this section, we present a method for analyzing the dynamics of robot manipulators known as the **Newton-Euler formulation**. This method leads to exactly the same final answers as the Lagrangian formulation presented in earlier sections, but the route taken is quite different. In particular, in the Lagrangian formulation we treat the manipulator as a whole and perform the analysis using a Lagrangian function (the difference between the kinetic energy and the potential energy). In contrast, in the Newton-Euler formulation we treat each link of the robot in turn, and write down the equations describing its linear motion and its angular motion. Of course, since each link is coupled to other links, these equations that describe each link contain coupling forces and torques that appear also in the equations that describe neighboring links. By doing a so-called forward-backward recursion, we are able to determine all of these coupling terms and eventually to arrive at a description of the manipulator as a whole. Thus we see that the philosophy of the Newton-Euler formulation is quite different from that of the Lagrangian formulation.

At this stage the reader can justly ask whether there is a need for another formulation, and the answer is not clear. Historically, both formulations were evolved in parallel, and each was perceived as having certain advantages. For instance, it was believed at one time that the Newton-Euler formulation is better suited to recursive computation than the

13.4 Advanced Mechanics Concepts

Newton-Euler mechanics has the advantage that it starts with very basic principles, but it has frustrating restrictions that make modeling more difficult for complicated mechanical systems. One of the main limitations is that all laws must be expressed in terms of an inertial frame with orthogonal axes. This section introduces the basic ideas of Lagrangian and Hamiltonian mechanics, which remove these restrictions by reducing mechanics to finding an optimal path using any coordinate neighborhood of the C-space. The optimality criterion is expressed in terms of energy. The resulting techniques can be applied on any coordinate neighborhood of a smooth manifold. The Lagrangian formulation is usually best for determining the motions of one or more bodies. Section 13.4.1 introduces the basic Lagrangian concepts based on the calculus of variations. Section 13.4.2 presents a general form of the Euler-Lagrange equations, which is useful for determining the motions of numerous dynamical systems, including chains of bodies. The Lagrangian is also convenient for systems that involve additional differential constraints, such as friction or rolling wheels. These cases are briefly covered in Section 13.4.3. The Hamiltonian formulation in Section 13.4.4 is based on a special phase space and provides an alternative to the Lagrangian formulation. The technique generalizes to Pontryagin's minimum principle, a powerful optimal control technique that is covered in Section 15.2.3.

13.4.1 Lagrangian Mechanics

Calculus of variations

Lagrangian mechanics is based on the *calculus of variations*, which is the subject of optimization over a space of paths. One of the most famous variational problems involves constraining a particle to travel along a curve (imagine that the particle slides along a frictionless track). The problem is to find the curve for which the ball travels from one point to the other, starting at rest, and being accelerated only by gravity. The solution is a cycloid function called the *Brachistochrone curve* [65]. Before this problem is described further, recall the classical optimization problem from calculus in which the task is to find extremal values (minima and maxima) of a function. Let $\tilde{x}$ denote a smooth function from $\mathbb{R}$ to $\mathbb{R}$, and let $x(t)$ denote its value for any $t \in \mathbb{R}$. From standard calculus, the extremal values of $\tilde{x}$ are all $t \in \mathbb{R}$ for which $\dot{x} = 0$. Suppose that at some $t' \in \mathbb{R}$, $\tilde{x}$ achieves a local minimum. To serve as a local minimum, tiny perturbations of t' should result in larger function values. Thus, there exists some $d > 0$ such that $x(t' + \epsilon) > x(t')$ for any $\epsilon \in [-d, d]$. Each ϵ represents a possible perturbation of t'.

The calculus of variations addresses a harder problem in which optimization occurs over a space of functions. For each function, a value is assigned by a criterion called a *functional*.[10] A procedure analogous to taking the derivative

[10] This is the reason why a cost *functional* has been used throughout the book. It is a function on a space of functions.

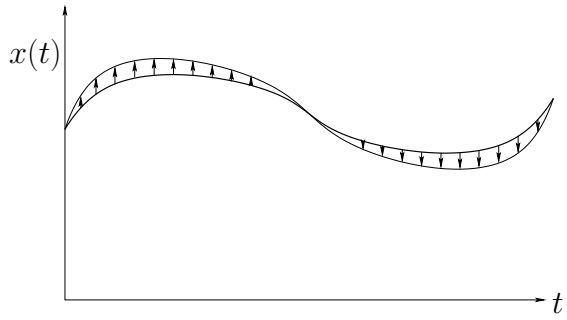

Figure 13.12: The variation is a "small" function that is added to $\tilde{x}$ to perturb it.

of the function and setting it to zero will be performed. This will be arrived at by considering tiny perturbations of an entire function, as opposed to the ϵ perturbations mentioned above. Each perturbation is itself a function, which is called a *variation*. For a function to minimize a functional, any small enough perturbation of it must yield a larger functional value. In the case of optimizing a function of one variable, there are only two directions for the perturbation: $\pm\epsilon$. See Figure 13.12. In the calculus of variations, there are many different "directions" because of the uncountably infinite number of ways to construct a small variation function that perturbs the original function (the set of all variations is an infinite-dimensional function space; recall Example 8.5).

Let $\tilde{x}$ denote a smooth function from $T = [t_0, t_1]$ into $\mathbb{R}$. The functional is defined by integrating a function over the domain of $\tilde{x}$. Let L be a smooth, real-valued function of three variables, a, b, and c.[11] The arguments of L may be any $a, b \in \mathbb{R}$ and $c \in T$ to yield $L(a, b, c)$, but each has a special interpretation. For some smooth function $\tilde{x}$, L is used to evaluate it at a particular $t \in T$ to obtain $L(x, \dot{x}, t)$. A *functional* Φ is constructed using L to evaluate the whole function $\tilde{x}$ as

$$\Phi(\tilde{x}) = \int_T L(x(t), \dot{x}(t), t) dt. \tag{13.114}$$

The problem is to select an $\tilde{x}$ that optimizes Φ. The approach is to take the derivative of Φ and set it equal to zero, just as in standard calculus; however, differentiating Φ with respect to $\tilde{x}$ is not standard calculus. This usually requires special conditions on the class of possible functions (e.g., smoothness) and on the vector space of variations, which are implicitly assumed to hold for the problems considered in this section.

[11] Unfortunately, L is used here to represent a cost function, on which a functional Φ will be based. This conflicts with using l as a cost function and L as the functional in motion planning formulations. This notational collision remains because L is standard notation for the Lagrangian. Be careful to avoid confusion.

Example 13.9 (Shortest-Path Functional) As an example of a functional, consider

$$L(x, \dot{x}, t) = \sqrt{1 + \dot{x}^2}. \tag{13.115}$$

When evaluated on a function $\tilde{x}$, this yields the arc length of the path. ■

Another example of a functional has already been seen in the context of motion planning. The cost functional (8.39) assigns a cost to a path taken through the state space. This provided a natural way to formulate optimal path planning. A discrete, approximate version was given by (7.26).

Let h be a smooth function over T, and let $\epsilon \in \mathbb{R}$ be a small constant. Consider the function defined as $x(t)+\epsilon h(t)$ for all $t \in [0, 1]$. If $\epsilon = 0$, then (13.114) remains the same. As ϵ is increased or decreased, then $\Phi(\tilde{x}+\epsilon h)$ may change. The function h is like the "direction" in a directional derivative. If for any smooth function h, their exists some $\epsilon > 0$ such that the value $\Phi(\tilde{x} + \epsilon h)$ increases, then $\tilde{x}$ is called an *extremal* of Φ. Any small perturbation to $\tilde{x}$ causes the value of Φ to increase. Therefore, $\tilde{x}$ behaves like a local minimum in a standard optimization problem.

Let $g = \epsilon h$ for some $\epsilon > 0$ and function h. The differential of a functional can be approximated as [6]

$$\begin{aligned}
\Phi(\tilde{x} + g) - \Phi(\tilde{x}) &= \int_T \Big(L(x(t) + g(t), \dot{x}(t) + \dot{g}(t), t) - L(x(t), \dot{x}(t), t) \Big) dt + \cdots \\
&= \int_T \left(\frac{\partial L}{\partial x} g + \frac{\partial L}{\partial \dot{x}} \dot{g} \right) dt + \cdots \\
&= \int_T \left(\frac{\partial L}{\partial x} g - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} g \right) dt + \left(\frac{\partial L}{\partial \dot{x}} g \right) \Bigg|_{t_0}^{t_1} + \cdots,
\end{aligned} \tag{13.116}$$

in which $\cdots$ represents higher order terms that will vanish in the limit. The last step follows from integration by parts:

$$\left(\frac{\partial L}{\partial \dot{x}} g \right) \Bigg|_{t_0}^{t_1} = \int_T \frac{\partial L}{\partial \dot{x}} \dot{g} dt + \int_T \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} h dt, \tag{13.117}$$

which is just $uv = \int v du + \int u dv$. Consider the value of (13.116) as ϵ becomes small, and assume that $h(t_0) = h(t_1) = 0$. For $\tilde{x}$ to be an extremal function, the change expressed in (13.116) should tend to zero as the variations approach zero. Based on further technical assumptions, including the *Fundamental Lemma of the Calculus of Variations* (see Section 12 of [6]), the *Euler-Lagrange equation*,

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0, \tag{13.118}$$

is obtained as a necessary condition for $\tilde{x}$ to be an extremum. Intuition can be gained by studying the last line of (13.116). The integral attains a zero value

precisely when (13.118) is satisfied. The other terms vanish because $h(t_0) = h(t_1) = 0$, and higher order terms disappear in the limit process.

The partial derivatives of L with respect to $\dot{x}$ and x are defined using standard calculus. The derivative $\partial L/\partial \dot{x}$ is evaluated by treating $\dot{x}$ as an ordinary variable (i.e., as $\partial L/\partial b$ when the variables are named as in $L(a, b, c)$). Following this, the derivative of $\partial L/\partial \dot{x}$ with respect to t is taken. To illustrate this process, consider the following example.

Example 13.10 (A Simple Variational Problem) Let L be a functional defined as

$$L(x, \dot{x}, t) = x^3 + \dot{x}^2. \tag{13.119}$$

The partial derivatives with respect to x and $\dot{x}$ are

$$\frac{\partial L}{\partial x} = 3x^2 \tag{13.120}$$

and

$$\frac{\partial L}{\partial \dot{x}} = 2\dot{x}. \tag{13.121}$$

Taking the time derivative of (13.121) yields

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 2\ddot{x} \tag{13.122}$$

Substituting these into the Euler-Lagrange equation (13.118) yields

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 2\ddot{x} - 3x^2 = 0. \tag{13.123}$$

This represents a second-order differential constraint that constrains the acceleration as $\ddot{x} = 3x^2/2$. By constructing a 2D phase space, the constraint could be expressed using first-order differential equations. ■

Hamilton's principle of least action

Now sufficient background has been given to return to the dynamics of mechanical systems. The path through the C-space of a system of bodies can be expressed as the solution to a calculus of variations problem that optimizes the difference between kinetic and potential energy. The calculus of variations principles generalize to any coordinate neighborhood of $\mathcal{C}$. In this case, the Euler-Lagrange equation is

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0, \tag{13.124}$$

in which q is a vector of n coordinates. It is actually n scalar equations of the form

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0. \tag{13.125}$$

The coming presentation will use (13.124) to obtain a phase transition equation. This will be derived by optimizing a functional defined as the change in kinetic and potential energy. Kinetic energy for particles and rigid bodies was defined in Section 13.3.1. In general, the kinetic energy function must be a quadratic function of $\dot{q}$. Its definition can be interpreted as an inner product on $\mathcal{C}$, which causes $\mathcal{C}$ to become a *Riemannian manifold* [19]. This gives the manifold a notion of the "angle" between velocity vectors and leads to well-defined notions of curvature and shortest paths called *geodesics*. Let $K(q, \dot{q})$ denote the kinetic energy, expressed using the manifold coordinates, which always takes the form

$$K(q, \dot{q}) = \tfrac{1}{2} \dot{q}^T M(q) \dot{q}, \tag{13.126}$$

in which $M(q)$ is an $n \times n$ matrix called the *mass matrix* or *inertia matrix*.

The next step is to define potential energy. A system is called *conservative* if the forces acting on a point depend only on the point's location, and the work done by the force along a path depends only on the endpoints of the path. The total energy is conserved under the motion of a conservative system. In this case, there exists a *potential function* $\phi : W \to \mathbb{R}$ such that $F = \partial\phi/\partial p$, for any $p \in W$. Let $V(q)$ denote the total *potential energy* of a collection of bodies, placed at configuration q.

It will be assumed that the dynamics are time-invariant. *Hamilton's principle of least action* states that the trajectory, $\tilde{q} : T \to \mathcal{C}$, of a mechanical system coincides with extremals of the functional,

$$\Phi(\tilde{q}) = \int_T \Big(K(q(t), \dot{q}(t)) - V(q(t)) \Big) dt, \tag{13.127}$$

using *any* coordinate neighborhood of $\mathcal{C}$. The principle can be seen for the case of $\mathcal{C} = \mathbb{R}^3$ by expressing Newton's second law in a way that looks like (13.124) [6]:

$$\frac{d}{dt}(m\dot{q}) - \frac{\partial V}{\partial q} = 0, \tag{13.128}$$

in which the force is replaced by the derivative of potential energy. This suggests applying the Euler-Lagrange equation to the functional

$$L(q, \dot{q}) = K(q, \dot{q}) - V(q), \tag{13.129}$$

in which it has been assumed that the dynamics are time-invariant; hence, $L(q, \dot{q}, t) = L(q, \dot{q})$. Applying the Euler-Lagrange equation to (13.127) yields the extremals.

The advantage of the Lagrangian formulation is that the C-space does not have to be $\mathcal{C} = \mathbb{R}^3$, described in an inertial frame. The Euler-Lagrange equation gives a necessary condition for the motions in any C-space of a mechanical system. The conditions can be expressed in terms of any coordinate neighborhood, as opposed to orthogonal coordinate systems, which are required by the Newton-Euler formulation. In mechanics literature, the q variables are often referred

to as *generalized coordinates*. This simply means the coordinates given by any coordinate neighborhood of a smooth manifold.

Thus, the special form of (13.124) that uses (13.129) yields the appropriate constraints on the motion:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial K(q, \dot{q})}{\partial \dot{q}} - \frac{\partial K(q, \dot{q})}{\partial q} + \frac{\partial V(q)}{\partial q} = 0. \tag{13.130}$$

Recall that this represents n equations, one for each coordinate q_i. Since $K(q, \dot{q})$ does not depend on time, the d/dt operator simply replaces $\dot{q}$ by $\ddot{q}$ in the calculated expression for $\partial K(q, \dot{q})/\partial \dot{q}$. The appearance of $\ddot{q}$ seems appropriate because the resulting differential equations are second-order, which is consistent with Newton-Euler mechanics.

Example 13.11 (A Falling Particle) Suppose that a particle with mass m is falling in $\mathbb{R}^3$. Let (q_1, q_2, q_3) denote the position of the particle. Let g denote the acceleration constant of gravity in the $-q_3$ direction. The potential energy is $V(q) = mgq_3$. The kinetic energy is

$$K(q, \dot{q}) = \tfrac{1}{2} m\dot{q} \cdot \dot{q} = \tfrac{1}{2} m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2). \tag{13.131}$$

The Lagrangian is

$$L(q, \dot{q}) = K(q, \dot{q}) - V(q) = \tfrac{1}{2} m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - mgq_3 = 0. \tag{13.132}$$

To obtain the differential constraints on the motion of the particle, use (13.130). For each i from 1 to 3,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{d}{dt}(m\dot{q}_i) = m\ddot{q}_i \tag{13.133}$$

Since $K(q, \dot{q})$ does not depend on q, the derivative $\partial K/\partial q_i = 0$ for each i. The derivatives with respect to potential energy are

$$\frac{\partial V}{\partial q_1} = 0 \qquad \frac{\partial V}{\partial q_2} = 0 \qquad \frac{\partial V}{\partial q_3} = mg. \tag{13.134}$$

Substitution into (13.130) yields three equations:

$$m\ddot{q}_1 = 0 \qquad m\ddot{q}_2 = 0 \qquad m\ddot{q}_3 + mg = 0. \tag{13.135}$$

These indicate that acceleration only occurs in the $-q_3$ direction, and this is due to gravity. The equations are consistent with Newton's laws. As usual, a six-dimensional phase space can be defined to obtain first-order differential constraints. ■

The "least" part of Hamilton's principle is actually a misnomer. It is technically only a principle of "extremal" action because (13.130) can also yield motions that maximize the functional.

Applying actions

Up to this point, it has been assumed that no actions are applied to the mechanical system. This is the way the Euler-Lagrange equation usually appears in physics because the goal is to predict motion, rather than control it. Let $u \in \mathbb{R}^n$ denote an action vector. Actions can be applied to the Lagrangian formulation as *generalized forces* that "act" on the right side of the Euler-Lagrange equation. This results in

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = u. \tag{13.136}$$

The actions force the mechanical system to deviate from its usual behavior. In some instances, the true actions may be expressed in terms of other variables, and then u is obtained by a transformation (recall transforming action variables for the differential drive vehicle of Section 13.1.2). In this case, u may be replaced in (13.136) by $\phi(u)$ for some transformation ϕ. In this case, the dimension of u need not be n.

Procedure for deriving the state transition equation

The following general procedure can be followed to derive the differential model using Lagrangian mechanics on a coordinate neighborhood of a smooth n-dimensional manifold:

1. Determine the degrees of freedom of the system and define the appropriate n-dimensional smooth manifold $\mathcal{C}$.

2. Express the kinetic energy as a quadratic form in the configuration velocity components:

$$K(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}(q)\dot{q}_i\dot{q}_j. \tag{13.137}$$

3. Express the potential energy $V(q)$.

4. Let $L(q, \dot{q}) = K(q, \dot{q}) - V(q)$ be the Lagrangian function, and use the Euler-Lagrange equation (13.130) to determine the differential constraints.

5. Convert to phase space form by letting $x = (q, \dot{q})$. If possible, solve for $\dot{x}$ to obtain $\dot{x} = f(x, u)$.

Example 13.12 (2D Rigid Body Revisited) The equations in (13.109) can be alternatively derived using the Euler-Lagrange equation. Let $\mathcal{C} = \mathbb{R}^2 \times \mathbb{S}^1$, and let $(q_1, q_2, q_3) = (x, y, \theta)$ to conform to the notation used to express the Lagrangian.

The kinetic energy is the sum of kinetic energies due to linear and angular velocities, respectively. This yields

$$K(q, \dot{q}) = \tfrac{1}{2}m\dot{q} \cdot \dot{q} + \tfrac{1}{2}I\dot{q}_3^2, \tag{13.138}$$

in which m and I are the mass and moment of inertia, respectively. Assume there is no gravity; hence, $V(q) = 0$ and $L(q, \dot{q}) = K(q, \dot{q})$.

Suppose that generalized forces u_1, u_2, and u_3 can be applied to the configuration variables. Applying the Euler-Lagrange equation to $L(q, \dot{q})$ yields

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} &= \frac{d}{dt}(m\dot{q}_1) = m\ddot{q}_1 = u_1 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} &= \frac{d}{dt}(m\dot{q}_2) = m\ddot{q}_2 = u_2 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_3} - \frac{\partial L}{\partial q_3} &= \frac{d}{dt}(I\dot{q}_3) = I\ddot{q}_3 = u_3.
\end{aligned} \tag{13.139}$$

These expressions are equivalent to those given in (13.109). One difference is that conversion to the phase space is needed. The second difference is that the action variables in (13.139) do not refer directly to forces and moments. They are instead interpreted as generalized forces that act on the configuration variables. A conversion should be performed if the original actions in (13.109) are required. ■

13.4.2 General Lagrangian Expressions

As more complicated mechanics problems are considered, it is convenient to express the differential constraints in a general form. For example, evaluating (13.130) for a kinematic chain of bodies leads to very complicated expressions. The terms of these expressions, however, can be organized into standard forms that appear simpler and give some intuitive meanings to the components.

Suppose that the kinetic energy is expressed using (13.126), and let $m_{ij}(q)$ denote an entry of $M(q)$. Suppose that the potential energy is $V(q)$. By performing the derivatives expressed in (13.136), the Euler-Lagrange equation can be expressed as n scalar equations of the form [68]

$$\sum_{j=1}^{n} m_{ij}(q)\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} h_{ijk}(q)\dot{q}_j\dot{q}_k + g_i(q) = u_i \tag{13.140}$$

in which

$$h_{ijk} = \frac{\partial m_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial m_{jk}}{\partial q_i}. \tag{13.141}$$

There is one equation for each i from 1 to n. The components of (13.140) have physical interpretations. The m_{ii} coefficients represent the inertia with respect to q_i. The m_{ij} represent the affect on q_j of accelerating q_i. The $h_{ijj}\dot{q}_j^2$ terms represent the centrifugal effect induced on q_i by the velocity of q_j. The $h_{ijk}\dot{q}_j\dot{q}_k$ terms represent the Coriolis effect induced on q_i by the velocities of q_j and q_k. The g_i term usually arises from gravity.

An alternative to (13.140) is often given in terms of matrices. It can be shown that the Euler-Lagrange equation reduces to

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u, \tag{13.142}$$

which represents n scalar equations. This introduces $C(q,\dot{q})$, which is an $n \times n$ *Coriolis matrix*. It turns out that many possible Coriolis matrices may produce equivalent different constraints. With respect to (13.140), the Coriolis matrix must be chosen so that

$$\sum_{j=1}^{n} c_{ij}\dot{q}_j = \sum_{j=1}^{n}\sum_{k=1}^{n} h_{ijk}\dot{q}_j\dot{q}_k. \tag{13.143}$$

Using (13.141),

$$\sum_{j=1}^{n} c_{ij}\dot{q}_j = \sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial m_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial m_{jk}}{\partial q_i}\right)\dot{q}_j\dot{q}_k. \tag{13.144}$$

A standard way to determine $C(q,\dot{q})$ is by computing *Christoffel symbols*. By subtracting $\frac{1}{2}\frac{\partial m_{jk}}{\partial q_i}$ from the inside of the nested sums in (13.144), the equation can be rewritten as

$$\sum_{j=1}^{n} c_{ij}\dot{q}_j = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial m_{ij}}{\partial q_k}\dot{q}_j\dot{q}_k + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial m_{ij}}{\partial q_k} - \frac{\partial m_{jk}}{\partial q_i}\right)\dot{q}_j\dot{q}_k. \tag{13.145}$$

This enables an element of $C(q,\dot{q})$ to be written as

$$c_{ij} = \sum_{k=1}^{n} c_{ijk}\dot{q}_k, \tag{13.146}$$

in which

$$c_{ijk} = \frac{1}{2}\left(\frac{\partial m_{ij}}{\partial q_k} + \frac{\partial m_{ik}}{\partial q_j} - \frac{\partial m_{jk}}{\partial q_i}\right). \tag{13.147}$$

This is called a *Christoffel symbol*, and it is obtained from (13.145). Note that $c_{ijk} = c_{ikj}$. Christoffel symbols arise in the study of affine connections in differential geometry and are usually denoted as Γ^i_{jk}. Affine connections provide a way to express acceleration without coordinates, in the same way that the tangent space was expressed without coordinates in Section 8.3.2. For affine connections in differential geometry, see [17]; for their application to mechanics, see [19].

Conversion to a phase transition equation

The final step is to convert the equations into phase space form. A $2n$-dimensional phase vector is introduced as $x = (q,\dot{q})$. The task is to obtain $\dot{x} = f(x,u)$, which

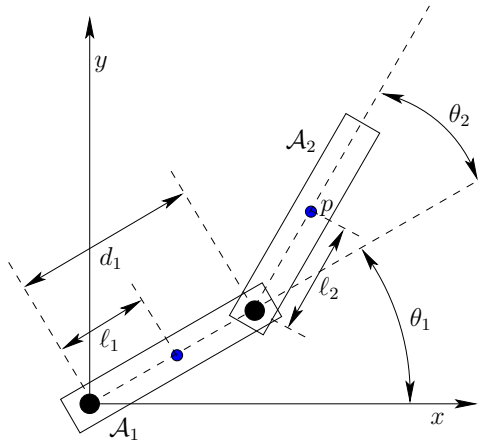

Figure 13.13: Parameter values for a two-link robot with two revolute joints.

represents $2n$ scalar equations. The first n equations are $\dot{x}_i = x_{n+i}$ for i from 1 to n. The final n equations are obtained by solving for $\ddot{q}$.

Suppose that the general form in (13.142) is used. Solving for $\ddot{q}$ yields

$$\ddot{q} = M(q)^{-1}(u - C(q,\dot{q})\dot{q} - g(q)). \tag{13.148}$$

The phase variables are then substituted in a straightforward manner. Each $\ddot{q}_i$ for i from 1 to n becomes $\dot{x}_{n+i}$, and $M(q)$, $C(q,\dot{q})$, and $g(q)$ are expressed in terms of x. This completes the specification of the phase transition equation.

Example 13.13 (Two-Link Manipulator) Figure 13.13 shows a two-link manipulator for which there are two revolute joints and two links, $\mathcal{A}_1$ and $\mathcal{A}_2$. Hence, $\mathcal{C} = \mathbb{S}^1 \times \mathbb{S}^1$. Let $q = (\theta_1, \theta_2)$ denote a configuration. Each of the two joints is controlled by a motor that applies a torque u_i. Let u_1 apply to the base, and let u_2 apply to the joint between $\mathcal{A}_1$ and $\mathcal{A}_2$. Let d_1 be the link length of $\mathcal{A}_1$. Let ℓ_i be the distance from the $\mathcal{A}_i$ origin to its center of mass. For each $\mathcal{A}_i$, let m_i and I_i be its mass and moment of inertia, respectively.

The kinetic energy of $\mathcal{A}_1$ is

$$K_1(\dot{q}) = \tfrac{1}{2}m_1\ell_1\dot{\theta}_1^2 + \tfrac{1}{2}I_1\dot{\theta}_1^2, \tag{13.149}$$

and the potential energy of $\mathcal{A}_1$ is

$$V_1(q) = m_1 g \ell_1 \sin\theta_1. \tag{13.150}$$

The kinetic energy of $\mathcal{A}_2$ is

$$K_2(\dot{q}) = \tfrac{1}{2}p \cdot p + \tfrac{1}{2}I_2(\dot{\theta}_1 + \dot{\theta}_2)^2, \tag{13.151}$$

in which p denotes the position of the center of mass of $\mathcal{A}_1$ and is given from (3.53) as

$$\begin{aligned} p_1 &= d_1 \cos\theta_1 + \ell_2 \cos\theta_2 \\ p_2 &= d_1 \sin\theta_1 + \ell_2 \sin\theta_2. \end{aligned} \tag{13.152}$$

The potential energy of $\mathcal{A}_2$ is

$$V_2(q) = m_2 g (d_1 \sin\theta_1 + \ell_2 \sin\theta_2). \tag{13.153}$$

At this point, the Lagrangian function can be formed as

$$L(q, \dot{q}) = K_1(\dot{\theta}_1) + K_2(\dot{\theta}_1, \dot{\theta}_2) - V_1(\theta_1) - V_2(\theta_1, \theta_2) \tag{13.154}$$

and inserted into (13.118) to obtain the differential constraints in implicit form, expressed in terms of $\ddot{q}$, $\dot{q}$, and q. Conversion to the phase space is performed by solving the implicit constraints for $\ddot{q}$ and assigning $x = (q, \dot{q})$, in which x is a four-dimensional phase vector.

Rather than performing the computations directly using (13.118), the constraints can be directly determined using (13.140). The terms are

$$M(q) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \tag{13.155}$$

in which

$$\begin{aligned} m_{11} &= I_1 + m_1 \ell_1^2 + I_2 + m_2 (d_1^2 + \ell_2^2 + 2 d_1 \ell_2 \cos\theta_2) \\ m_{12} &= m_{21} = I_2 + m_2 (\ell_2^2 + d_1 \ell_2 \cos\theta_2) \\ m_{22} &= I_2 + m_2 \ell_2^2, \end{aligned} \tag{13.156}$$

and

$$\begin{aligned} c_{111} &= \frac{1}{2} \frac{\partial m_{11}}{\partial \theta_1} = 0 \\ c_{112} = c_{121} &= \frac{1}{2} \frac{\partial m_{11}}{\partial \theta_2} = -m_2 \ell_1 \ell_2 p_2 \\ c_{122} &= \frac{\partial m_{12}}{\partial \theta_2} - \frac{1}{2} \frac{\partial m_{22}}{\partial \theta_1} = -m_2 \ell_1 \ell_2 p_2 \\ c_{211} &= \frac{\partial m_{21}}{\partial \theta_1} - \frac{1}{2} \frac{\partial m_{11}}{\partial \theta_2} = m_2 \ell_1 \ell_2 p_2 \\ c_{212} = c_{221} &= \frac{1}{2} \frac{\partial m_{22}}{\partial \theta_1} = 0 \\ c_{222} &= \frac{1}{2} \frac{\partial m_{22}}{\partial \theta_2} = 0. \end{aligned} \tag{13.157}$$

The final term is defined as

$$\begin{aligned} g_1 &= (m_1 \ell_1 + m_2 d_1) g p_1 + m_1 \ell_2 p_2 \\ g_2 &= m_2 \ell_2 g p_2. \end{aligned} \tag{13.158}$$

The dynamics can alternatively be expressed using $M(q)$, $C(q, \dot{q})$, and $g(q)$ in (13.142). The Coriolis matrix is defined using (13.143) to obtain

$$C(q, \dot{q}) = -m_2 \ell_1 \ell_2 p_2 \begin{pmatrix} \dot{\theta}_2 & \dot{\theta}_1 + \dot{\theta}_2 \\ \dot{\theta}_1 & 0 \end{pmatrix}, \tag{13.159}$$

in which p_2 is defined in (13.152) and is a function of q. For convenience, let

$$r = m_2 \ell_1 \ell_2 p_2. \tag{13.160}$$

The resulting expression, which is now a special form of (13.142), is

$$\begin{aligned} m_{11} \ddot{\theta}_1 + m_{12} \ddot{\theta}_2 - 2 r \dot{\theta}_1 \dot{\theta}_2 - r \dot{\theta}_2^2 + g_1(q) &= u_1 \\ m_{22} \ddot{\theta}_1 + m_{21} \ddot{\theta}_2 + r \dot{\theta}_1^2 + g_2(q) &= u_2. \end{aligned} \tag{13.161}$$

The phase transition equation is obtained by letting $x = (\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$ and substituting the state variables into (13.161). The variables $\ddot{\theta}_1$ and $\ddot{\theta}_2$ become $\dot{x}_3$ and $\dot{x}_4$, respectively. The equations must be solved for $\dot{x}_3$ and $\dot{x}_4$. An extension of this model to motors that have gear ratios and nonnegligible mass appears in [68]. ■

The example provided here barely scratches the surface on the possible systems that can be elegantly modeled. Many robotics texts cover cases in which there are more links, different kinds of joints, and frictional forces [29, 56, 68, 70, 79].

The phase transition equation for chains of bodies could alternatively be derived using the Newton-Euler formulation of mechanics. Even though the Lagrangian form is more elegant, the Newton-Euler equations, when expressed recursively, are far more efficient for simulations of multibody dynamical systems [29, 69, 79].

13.4.3 Extensions of the Euler-Lagrange Equations

Several extensions of the Euler-Lagrange equation can be constructed to handle complications that arise in addition to kinetic energy and potential energy in a conservative field. Each extension usually involves adding more terms to (13.129) to account for the new complication. Problems that can be handled in this way are closed kinematic chains, nonholonomic constraints, and nonconservative forces (such as friction).

Incorporating velocity constraints

The Lagrangian formulation of Section 13.4.1 can be extended to allow additional constraints placed on q and $\dot{q}$. This is very powerful for developing state transition equations for robots that have closed kinematic chains or wheeled bodies. If there

are closed chains, then the configurations may be restricted to lie in a subset of $\mathcal{C}$. If a parameterization of the solution set is possible, then $\mathcal{C}$ can be redefined over the reduced C-space. This is usually not possible, however, because such a parametrization is difficult to obtain, as mentioned in Section 4.4. If there are wheels or other contact-based constraints, such as those in Section 13.1.3, then extra constraints on q and $\dot{q}$ exist. Dynamics can be incorporated into the models of Section 13.1 by extending the Euler-Lagrange equation.

The coming method will be based on Lagrange multipliers. Recall from standard calculus that to optimize a function h defined over $\mathbb{R}^n$, subject to an implicit constraint $g(x) = 0$, it is sufficient to consider only the extrema of

$$h(x) + \lambda g(x), \tag{13.162}$$

in which $\lambda \in \mathbb{R}$ represents a *Lagrange multiplier* [40]. The extrema are found by solving

$$\nabla h(x) + \lambda \nabla g(x) = 0, \tag{13.163}$$

which expresses n equations of the form

$$\frac{\partial h}{\partial x_i} + \lambda \frac{\partial g}{\partial x_i} = 0. \tag{13.164}$$

The same principle applies for handling velocity constraints on $\mathcal{C}$.

Suppose that there are velocity constraints on $\mathcal{C}$ as considered in Section 13.1. Consider implicit constraints, in which there are k equations of the form $g_i(q, \dot{q}) = 0$ for i from 1 to k. Parametric constraints can be handled as a special case of implicit constraints by writing

$$g_i(q, \dot{q}) = \dot{q}_i - f_i(q, u) = 0. \tag{13.165}$$

For any constraints that contain actions u, no extra difficulties arise. Each u_i is treated as a constant in the following analysis. Therefore, action variables will not be explicitly named in the expressions.

As before, assume time-invariant dynamics (see [60] for the time-varying case). Starting with $L(q, \dot{q})$ defined using (13.130), let the new criterion be

$$L_c(q, \dot{q}, \lambda) = L(q, \dot{q}) + \sum_{i=1}^{k} \lambda_i g_i(q, \dot{q}). \tag{13.166}$$

A functional Φ_c is defined by substituting L_c for L in (13.114).

The extremals of Φ_c are given by n equations,

$$\frac{d}{dt}\frac{\partial L_c}{\partial \dot{q}_i} - \frac{\partial L_c}{\partial q_i} = 0, \tag{13.167}$$

and k equations,

$$\frac{d}{dt}\frac{\partial L_c}{\partial \dot{\lambda}_i} - \frac{\partial L_c}{\partial \lambda_i} = 0. \tag{13.168}$$

The justification for this is the same as for (13.124), except now λ is included. The equations of (13.168) are equivalent to the constraints $g_i(q, \dot{q}) = 0$. The first term of each is zero because $\dot{\lambda}$ does not appear in the constraints, which reduces them to

$$\frac{\partial L_c}{\partial \lambda_i} = 0. \tag{13.169}$$

This already follows from the constraints on extremals of L and the constraints $g_i(q, \dot{q}) = 0$. In (13.167), there are n equations in $n+k$ unknowns. The k Lagrange multipliers can be eliminated by using the k constraints $g_i(q, \dot{q}) = 0$. This corresponds to Lagrange multiplier elimination in standard constrained optimization [40].

The expressions in (13.167) and the constraints $g_i(q, \dot{q})$ may be quite complicated, which makes the determination of a state transition equation challenging. General forms are given in Section 3.8 of [60]. An important special case will be considered here. Suppose that the constraints are Pfaffian,

$$g_i(q, \dot{q}) = \sum_{j=1}^{n} g_{ij}(q)\dot{q}_j = 0, \tag{13.170}$$

as introduced in Section 13.1. This includes the nonholonomic velocity constraints due to wheeled vehicles, which were presented in Section 13.1.2. Furthermore, this includes the special case of constraints of the form $g_i(q) = 0$, which models closed kinematic chains. Such constraints can be differentiated with respect to time to obtain

$$\frac{d}{dt} g_i(q) = \sum_{j=1}^{n} \frac{\partial g_i}{\partial q_j}\dot{q}_j = \sum_{j=1}^{n} g_{ij}(q)\dot{q}_j = 0, \tag{13.171}$$

which is in the Pfaffian form. This enables the dynamics of closed chains, considered in Section 4.4, to be expressed without even having a parametrization of the subset of $\mathcal{C}$ that satisfies the closure constraints. Starting in implicit form, differentiation is required to convert them into the Pfaffian form.

For the important case of Pfaffian constraints, (13.167) simplifies to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + \sum_{j=1}^{k} \lambda_j g_{ji}(q) = 0, \tag{13.172}$$

The Pfaffian constraints can be used to eliminate the Lagrange multipliers, if desired. Note that g_{ji} represents the ith term of the jth Pfaffian constraint. An action variable u_i can be placed on the right side of each constraint, if desired.

Equation (13.172) often appears instead as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \sum_{l=1}^{k} \lambda_j g_{ji}(q, \dot{q}), \tag{13.173}$$

which is an alternative but equivalent expression of constraints because the Lagrange multipliers can be negated without affecting the existence of extremals. In this case, a nice interpretation due to D'Alembert can be given. Expressions that appear on the right of (13.173) can be considered as actions, as mentioned in Section 13.4.1. As stated previously, such actions are called generalized forces in mechanics. The *principle of virtual work* is obtained by integrating the reaction forces needed to maintain the constraints. These reaction forces are precisely given on the right side of (13.173). Due to the cancellation of forces, no true work is done by the constraints (if there is no friction).

Example 13.14 (A Particle on a Sphere) Suppose that a particle travels on a unit sphere without friction or gravity. Let $(q_1, q_2, q_3) \in \mathbb{R}^3$ denote the position of the point. The Lagrangian function is the kinetic energy,

$$L(q, \dot{q}) = \tfrac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2), \tag{13.174}$$

in which m is the particle mass. For simplicity, assume that $m = 2$.

The constraint that the particle must travel on a sphere yields

$$g_1(q) = q_1^2 + q_2^2 + q_3^2 - 1 = 0. \tag{13.175}$$

This can be put into Pfaffian form by time differentiation to obtain

$$2q_1\dot{q}_1 + 2q_2\dot{q}_2 + 2q_3\dot{q}_3 = 0. \tag{13.176}$$

Since $k = 1$, there is a single Lagrange multiplier λ_1. Applying (13.172) yields three equations,

$$\ddot{q}_i - 2q_i\lambda_1 = 0, \tag{13.177}$$

for i from 1 to 3. The generic form of the solution is

$$c_1q_1 + c_2q_2 + c_3q_3 = 0, \tag{13.178}$$

in which the c_i are real-valued constants that can be determined from the initial position of the particle. This represents the equation of a plane through the origin. The intersection of the plane with the sphere is a great circle. This implies that the particle moves between two points by traveling along the great circle. These are the shortest paths (geodesics) on the sphere. ■

The general forms in Section 13.4.2 can be extended to the constrained case. For example, (13.142) generalizes to

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) + G(q)^T\lambda = u, \tag{13.179}$$

in which G is a $n \times k$ matrix that represents all of the g_{ji} Pfaffian coefficients. In this case, the Lagrange multipliers can be computed as [56]

$$\lambda = \big(G(q)M(q)^{-1}G(q)^T\big)^{-1} G(q)M(q)^{-1}\big(u - C(q, \dot{q})\dot{q}\big), \tag{13.180}$$

assuming G is time-invariant.

The phase transition equation can be determined in the usual way by performing the required differentiations, defining the $2n$ phase variables, and solving for $\dot{x}$. The result generalizes (13.148).

Nonconservative forces

The Lagrangian formulation has been extended so far to handle constraints on $\mathcal{C}$ that lower the dimension of the tangent space. The formulation can also be extended to allow nonconservative forces. The most common and important example in mechanical systems is friction. The details of friction models will not be covered here; see [52]. As examples, friction can arise when bodies come into contact, as in the joints of a robot manipulator, and as bodies move through a fluid, such as air or water. The nonconservative forces can be expressed as additional generalized forces, expressed in an $n \times 1$ vector of the form $B(q, \dot{q})$. Suppose that an action vector is also permitted. The modified Euler-Lagrange equation then becomes

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = u - B(\dot{q}, q). \tag{13.181}$$

A common extension to (13.142) is

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + N(q, \dot{q}) = u, \tag{13.182}$$

in which $N(q, \dot{q})$ generalizes $g(q)$ to include nonconservative forces. This can be generalized even further to include Pfaffian constraints and Lagrange multipliers,

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + N(q, \dot{q}) + G(q)^T\lambda = u. \tag{13.183}$$

The Lagrange multipliers become [56]

$$\lambda = \big(G(q)M(q)^{-1}G(q)^T\big)^{-1} G(q)M(q)^{-1}\big(u - C(q, \dot{q})\dot{q} - N(q, \dot{q})\big). \tag{13.184}$$

Once again, the phase transition equation can be derived in terms of $2n$ phase variables and generalizes (13.148).

13.4.4 Hamiltonian Mechanics

The Lagrangian formulation of mechanics is the most convenient for determining a state transition equation for a collection of bodies. Once the kinetic and potential energies are determined, the remaining efforts are straightforward computation of derivatives and algebraic manipulation. Hamiltonian mechanics provides an alternative formulation that is closely related to the Lagrangian. Instead of expressing second-order differential constraints on an n-dimensional C-space, it expresses first-order constraints on a $2n$-dimensional phase space. This idea should be familiar from Section 13.2. The new phase space considered here is