METR4202 -- Robotics
Tutorial 4 – Week 4: Solutions

Solutions updated by Chris, Jeevan, Russell. Thank you!

Reading
Please read/review chapter 8 & 9 of Robotics, Vision and Control.

Review: Forward Kinematics of a two-link planar manipulator

$$x = a_1 cos\theta_1 + a_2\cos(\theta_1 + \theta_2)$$
$$y = a_1 \sin\theta_1 + a_2 \sin(\theta_1 + \theta_2)$$

Questions

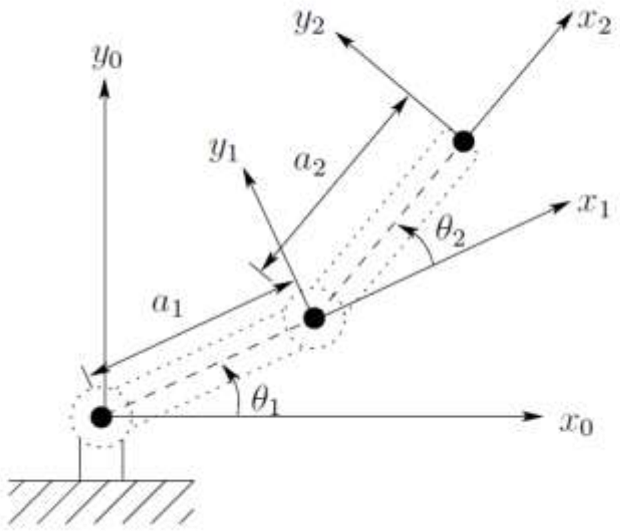

Figure 1: Two-link planar manipulator

1.

a.) Using the two-link planar manipulator from tutorial 3, calculate the jacobian needed to relate the joint velocities to the tool-point velocities.

$$J = \begin{bmatrix} -l_1 \sin(\theta_1) - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

b.) Similarly, calculate the inverse jacobian needed to relate the tool-point velocities to the joint velocities.

$$\begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \end{bmatrix} = \frac{1}{L_1 L_2 s_2} \begin{bmatrix} L_2 c_{12} & L_2 s_{12} \\ -L_1 c_1 - L_2 c_{12} & -L_1 s_1 - L_2 s_{12} \end{bmatrix} \begin{bmatrix} \dot x \\ \dot y \end{bmatrix}$$

2.

a.) Using the Jacobian found in Q1a, calculate the tool point linear velocity if joint 1 is rotating at 1 rad/s and joint 2 is rotating at 3 rad/s (a_1 = 2, a_2 = 3 θ_1= 167.028^o, θ_2 =-156.44^o).

$$\vec{v} = J\vec{\dot{q}}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1 \sin(\theta_1) - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -1.0000 & -0.5512 \\ 1.0000 & 2.9489 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -2.654 \\ 9.847 \end{bmatrix} m/s$$

b.) Calculate the resulting joint torques τ, given a force F = (30, -20) is applied to the end effector tool point.

$$\tau = J^T F$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -l_1 \sin(\theta_1) - l_2 \sin(\theta_1 + \theta_2) & l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) \\ -l_2 \sin(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -1.0000 & 1.0000 \\ -0.5512 & 2.9489 \end{bmatrix} \begin{bmatrix} 30 \\ -20 \end{bmatrix}$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -50.0 \\ -75.1 \end{bmatrix} Nm$$

3. (See also p. 209 of Spong, *Robot Modeling and Control* [p. 17 of attached PDF] or Ex 13.13 (p.637) of LaValle, *Planning Algorithms* [p.772 of Ch. 13 of the online PDF], or p. 110 of Asada and Slotine, *Robot Analysis and Control*)

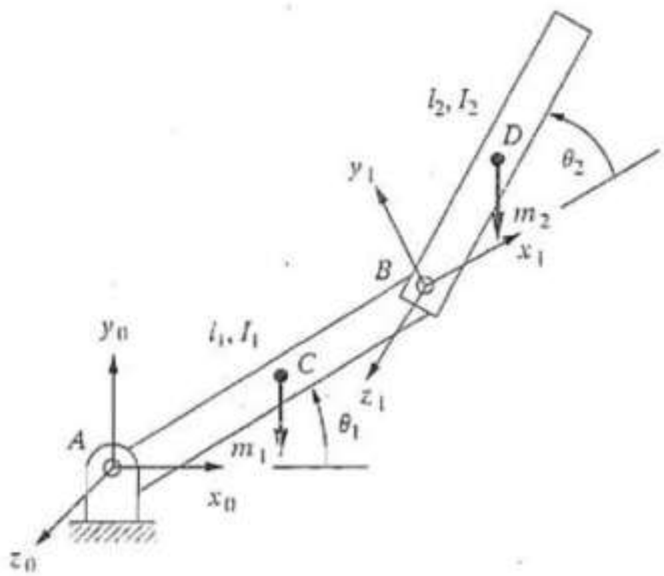

Figure 2: Two-link revolute joint arm.

a.) With respect to figure 2 above, derive the equations of motion for the two-degree-of-freedom robot arm using the Lagrangian method.

$$\tau_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta_\iota}}\right) - \frac{\partial L}{\partial \theta_i}$$

Step 1: Calculate the velocities at the center of mass of link 2.

$$x_D = l_1 C_1 + 0.5 l_2 C_{12} \;\; => \;\; \dot{x_D} = -l_1 S_1 \dot{\theta}_1 - 0.5 l_2 S_{12}(\dot{\theta}_1 + \dot{\theta}_2)$$

$$y_d = l_1 S_1 + 0.5 l_2 S_{12} \;\; => \;\; \dot{y_d} = l_1 C_1 \dot{\theta}_1 + 0.5 l_2 C_{12}(\dot{\theta}_1 + \dot{\theta}_2)$$

Total Velocity :

$$v_D{}^2 = \dot{x_D^2} + \dot{y_D^2}$$

$$= \dot{\theta}_1(l_1^2 + 0.25 l_2^2 + l_1 l_2 C_2) + \dot{\theta}_2^2(0.25 l_2^2) + \dot{\theta}_1 \dot{\theta}_2(0.5 l_2^2 + l_1 l_2 C_2)$$

Step 2: Calculate total Kinetic energy.

$$K = K_1 + K_2$$

$$= \left[\frac{1}{2} I_A \dot{\theta}_1^2\right] + \left[\frac{1}{2} I_D \left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + \frac{1}{2} m_2 v_d^2\right]$$

Substitute the total velocity into the Kinetic energy.

$$K = \dot{\theta}^2 \left(\frac{1}{6} m_1 l_1^2 + \frac{1}{6} m_2 l_2^2 + \frac{1}{2} m_2 l_1^2 + \frac{1}{2} m_2 l_1 l_2 C_2\right) + \dot{\theta}_2^2 \left(\frac{1}{6} m_2; l_2^2\right)$$

$$+ \dot{\theta}_1 \dot{\theta}_2 \left(\frac{1}{3} m_2 l_2^2 + \frac{1}{2} m_2 l_1 l_2 C_2\right)$$

Step 3: Calculate the total Potential energy of the system.

$$P = \frac{m_1 g l_1}{2} S_1 + m_2 g (l_1 S_1 + \frac{l_2}{2} S_{12})$$

The Lagrangian for the two-link robot arm is:

$$L = K - P$$

$$= \dot{\theta}^2\left(\frac{1}{6}m_1 l_1^2 + \frac{1}{6}m_2 l_2^2 + \frac{1}{2}m_2 l_1^2 + \frac{1}{2}m_2 l_1 l_2 C_2\right) + \dot{\theta}_2^2\left(\frac{1}{6}m_2; l_2^2\right)$$
$$+ \dot{\theta}_1\dot{\theta}_2\left(\frac{1}{3}m_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2 C_2\right) - \frac{m_1 g l_1}{2}S_1 - m_2 g(l_1 S_1 + \frac{l_2}{2}S_{12})$$

Step 4: Calculate the derivatives of the Lagrangian to determine the torque equations for the two-link robot arm: **Recall Chain rule expansion:**

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\frac{d}{dt} = \frac{\partial}{\partial \theta_2}\frac{d\theta_2}{dt} + \frac{\partial}{\partial \dot{\theta_1}}\frac{d\dot{\theta}_1}{dt}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta_1}}\right)$$
$$= \left(\frac{1}{3}m_1 l_1^2 + m_2 l_1^2 + \frac{1}{3}m_2 l_2^2 + m_2 l_1 l_2 C_2\right)\ddot{\theta_1}$$
$$+ \left(\frac{1}{3}m_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2 C_2\right)\ddot{\theta_2} - (m_2 l_1 l_2 S_2)\dot{\theta}_1\dot{\theta}_2 - \left(\frac{1}{2}m_2 l_1 l_2 S_2\right)\dot{\theta}_2^2$$

$$\frac{\partial L}{\partial \theta_1} = \left(\frac{1}{2}m_1 + m_2\right)g l_1 C_1 + \frac{1}{2}m_2 g l_2 C_{12}$$

$$\tau_1 = \left(\tfrac{1}{3}m_1 l_1^2 + m_2 l_1^2 + \tfrac{1}{3}m_2 l_2^2 + m_2 l_1 l_2 C_2\right)\ddot{\theta_1} + \left(\tfrac{1}{3}m_2 l_2^2 + \tfrac{1}{2}m_2 l_1 l_2 C_2\right)\ddot{\theta_2} - (m_2 l_1 l_2 S_2)\dot{\theta}_1\dot{\theta}_2 - \left(\tfrac{1}{2}m_2 l_1 l_2 S_2\right)\dot{\theta}_2^2 + \left(\tfrac{1}{2}m_1 + m_2\right)g l_1 C_1 + \tfrac{1}{2}m_2 g l_2 C_{12}$$

$$\frac{d}{dt} = \frac{\partial}{\partial \theta_2}\frac{d\theta_2}{dt} + \frac{\partial}{\partial \dot{\theta_2}}\frac{d\dot{\theta}_2}{dt}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta_2}}\right)$$
$$= \left(\frac{1}{3}m_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2 C_2\right)\ddot{\theta_1} + (\frac{1}{3}m_2 l_2^2)\ddot{\theta_2} - \left(\frac{1}{2}m_2 l_1 l_2 S_2\right)\dot{\theta}_1^2$$
$$- (m_2 l_1 l_2 S_2)\dot{\theta}_1\dot{\theta}_2$$

$$\frac{\partial L}{\partial \theta_2} = (m_2 l_1 l_2 S_2)\dot{\theta}_1\dot{\theta}_2 + \frac{1}{2}m_2 g l_2 C_{12}$$

$$\tau_2 = \left(\frac{1}{3}m_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2 C_2\right)\ddot{\theta_1} + \left(\frac{1}{3}m_2 l_2^2\right)\ddot{\theta_2} - \left(\frac{1}{2}m_2 l_1 l_2 S_2\right)\dot{\theta}_1^2 + \frac{1}{2}m_2 g l_2 C_{12}$$

Challenge Question:

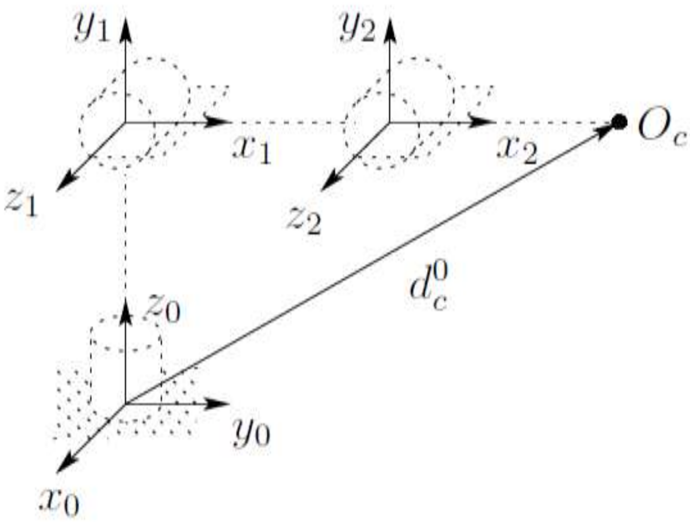

Figure 3: Elbow Manipulator

a.) List the DH parameters for this arm, clearly indication which parameters are the joint variables (L_1 = 3m. L_2 = 2m, L_3 = 1m).

b.) Find the inverse Kinematic equations for the arm to derive the joint values from tool point position.

c.) Given that the tool point is at $(1.0\text{m}, 0.2\text{m}, 0.5\text{m})^T$, use the inverse kinematic equations to find the joint values.

d.) Find the manipulator Jacobian, J, that relates the joint velocities to the tool point velocity.