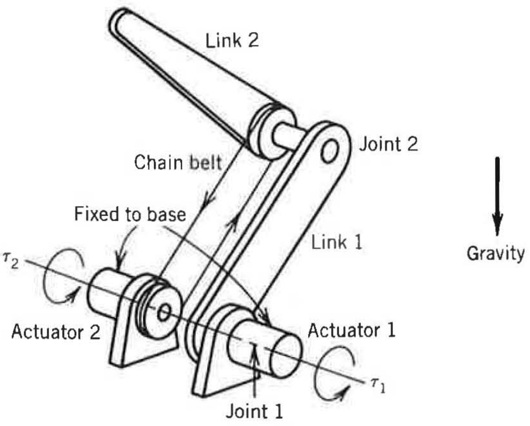

Figure 5-7 : Remotely driven two d.o.f. planar manipulator.

Note that, since no external force acts on the endpoint, the generalized forces coincide with the joint torques, as shown in equation (5-33). Equation (5-46) is the same as equation (5-11), which was derived from the Newton-Euler equations. $\Delta\Delta\Delta$

Example 5-3

Figure 5-7 shows a planar manipulator whose arm links have the same mass properties as those of the manipulator of Figure 5-2. The actuators and transmissions, however, are different. The second actuator, driving joint 2, is now located at the base, and the output torque is transmitted to joint 2 through a chain drive mechanism. Since the actuator is fixed to the base link, its reaction torque acts on the base link, while in Figure 5-2 the reaction torque of the second actuator acts on link 1. The first actuator, on the other hand, is the same for the two manipulators. Let us find Lagrange's equations of motion for this remotely driven manipulator.

The manipulator inertia tensor and the potential function are the same as for the manipulator of Figure 5-2. Let us investigate the virtual work done by the generalized forces. Letting τ_1^* and τ_2^* be the torques exerted by the first and the second actuators, respectively, the virtual work done by these torques is

$$\delta\text{Work} = \overset{*}{\tau}_1\,\delta\theta_1 + \overset{*}{\tau}_2\,(\delta\theta_1 + \delta\theta_2)$$
$$= (\overset{*}{\tau}_1 + \overset{*}{\tau}_2)\,\delta\theta_1 + \overset{*}{\tau}_2\,\delta\theta_2 \tag{5-47}$$

Comparing the above expression with (5-32):

$$\delta\text{Work} = \mathbf{Q}^T\,\delta\mathbf{q} = Q_1\delta q_1 + Q_2\delta q_2$$

where $\delta q_1 = \delta\theta_1$ and $\delta q_2 = \delta\theta_2$, we find that the generalized forces are

$$Q_1 = \overset{*}{\tau}_1 + \overset{*}{\tau}_2 \qquad\qquad Q_2 = \overset{*}{\tau}_2 \tag{5-48}$$

Replacing τ_1 and τ_2 in equation (5-46) by Q_1 and Q_2, respectively, we obtain the dynamic equations of the remotely driven manipulator. ΔΔΔ

5.2.4. Transformations of Generalized Coordinates

In the previous section, we used joint displacements as a complete set of independent generalized coordinates to describe Lagrange's equations of motion. However, any complete set of independent generalized coordinates can be used. It is a significant feature of the Lagrangian formulation that we can employ any convenient coordinates to describe the system. Also, in the Lagrangian formulation, coordinate transformations can be performed in a simple and systematic manner.

As before, let $\mathbf{q} = [q_1, \cdots, q_n]^T$ be the vector of joint coordinates, which represents a complete and independent set of generalized coordinates. We now assume that there exists another set of complete and independent generalized coordinates, $\mathbf{p} = [p_1, \cdots, p_n]^T$, that satisfy the following differential relationship with $\mathbf{q}$:

$$d\mathbf{p} = \mathbf{J}d\mathbf{q} \tag{5-49}$$

The Jacobian matrix $\mathbf{J}$ is assumed to be a non-singular square matrix within a specified region in $\mathbf{q}$-coordinates. Let us derive Lagrange's equations of motion in $\mathbf{p}$-coordinates from the ones

5.2. Lagrangian Formulation of Manipulator Dynamics

5.2.1. Lagrangian Dynamics

In the Newton-Euler formulation, the equations of motion are derived from Newton's Second Law, which relates force and momentum, as well as torque and angular momentum. The resulting equations involve constraint forces, which must be eliminated in order to obtain closed-form dynamic equations. In the Newton-Euler formulation, the equations are not expressed in terms of independent variables, and do not include input joint torques explicitly. Arithmetic operations are needed to derive the closed-form dynamic equations. This represents a complex procedure which requires physical intuition, as discussed in the previous section.

An alternative to the Newton-Euler formulation of manipulator dynamics is the Lagrangian formulation, which describes the behavior of a dynamic system in terms of work and energy stored in the system rather than of forces and momenta of the individual members involved. The constraint forces involved in the system are automatically eliminated in the formulation of Lagrangian dynamic equations. The closed-form dynamic equations can be derived systematically in any coordinate system.

Let $q_1, \cdots, q_n$ be generalized coordinates that completely locate a dynamic system. Let T and U be the total kinetic energy and potential energy stored in the dynamic system. We define the Lagrangian L by

$$L(q_i, \dot{q}_i) = T - U \tag{5-20}$$

Note that, since the kinetic and potential energies are functions of q_i and $\dot{q}_i$, $(i = 1, \cdots, n)$, so is the Lagrangian L. Using the Lagrangian, equations of motion of the dynamic system are given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i \qquad i=1, \cdots, n \tag{5-21}$$

where Q_i is the generalized force corresponding to the generalized coordinate q_i. The

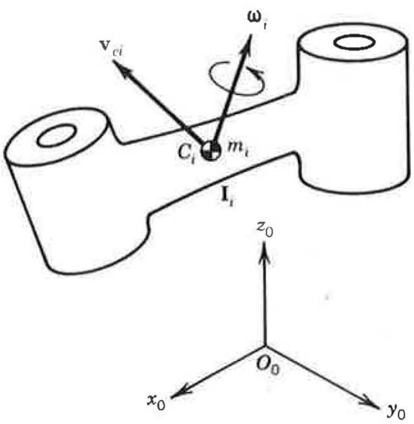

Figure 5-6 : Centroidal velocity and angular veloity of link i.

generalized force can be identified by considering the virtual work done by non-conservative forces acting on the system.

5.2.2. The Manipulator Inertia Tensor

In this section and the following section, we derive the equations of motion of a manipulator arm using the Lagrangian. We begin by deriving the kinetic energy stored in an individual arm link. As shown in Figure 5-6, let $\mathbf{v}_{ci}$ and ω_i be the 3 × 1 velocity vector of the centroid and the 3 × 1 angular velocity vector with reference to the base coordinate frame, which is an inertial reference frame. The kinetic energy of link i is then given by

$$T_i = \frac{1}{2} m_i \mathbf{v}_{ci}^T \mathbf{v}_{ci} + \frac{1}{2} \omega_i^T \mathbf{I}_i \omega_i \tag{5-22}$$

where m_i is the mass of the link and $\mathbf{I}_i$ is the 3 × 3 inertia tensor at the centroid expressed in the base coordinates. The first term in the above equation accounts for the kinetic energy resulting from the translational motion of the mass m_i, while the second term represents the kinetic energy resulting from rotation about the centroid. The total kinetic energy stored in the whole arm linkage is then given by

$$T = \sum_{i=1}^{n} T_i \tag{5-23}$$

since energy is additive.

The expression for the kinetic energy is written in terms of the velocity and angular velocity of each link member, which are not independent variables, as mentioned in the previous section. Let us now rewrite the above equations in terms of an independent and complete set of generalized coordinates, namely joint displacements $q = [q_1, \cdots, q_n]^T$. In Chapter 3, we analyzed the velocity and angular velocity of an end-effector in relation to joint velocities. We can employ the same method to compute the velocity and angular velocity of an individual link, if we regard the link as an end-effector. Namely, replacing subscripts n and e by i and ci, respectively, in equations (3-19) and (3-23), we obtain

$$
\begin{aligned}
\mathbf{v}_{ci} &= \mathbf{J}_{L1}^{(i)}\dot{q}_1 + \cdots + \mathbf{J}_{Li}^{(i)}\dot{q}_i = \mathbf{J}_{L}^{(i)}\dot{\mathbf{q}} \\
\boldsymbol{\omega}_i &= \mathbf{J}_{A1}^{(i)}\dot{q}_1 + \cdots + \mathbf{J}_{Ai}^{(i)}\dot{q}_i = \mathbf{J}_{A}^{(i)}\dot{\mathbf{q}}
\end{aligned}
\qquad (5\text{–}24)
$$

where $\mathbf{J}_{Lj}^{(i)}$ and $\mathbf{J}_{Aj}^{(i)}$ are the j-th column vectors of the $3 \times n$ Jacobian matrices $\mathbf{J}_{L}^{(i)}$ and $\mathbf{J}_{A}^{(i)}$, for linear and angular velocities of link i, respectively. Namely,

$$
\begin{aligned}
\mathbf{J}_{L}^{(i)} &= [\mathbf{J}_{L1}^{(i)} \ \cdots \ \mathbf{J}_{Li}^{(i)} \ \mathbf{0} \ \cdots \ \mathbf{0}] \\
\mathbf{J}_{A}^{(i)} &= [\mathbf{J}_{A1}^{(i)} \ \cdots \ \mathbf{J}_{Ai}^{(i)} \ \mathbf{0} \ \cdots \ \mathbf{0}]
\end{aligned}
\qquad (5\text{–}25)
$$

Note that, since the motion of link i depends on only joints 1 through i, the column vectors are set to zero for $j \geq i$. From equations (3-26) and (3-27) each column vector is given by

$$
\begin{aligned}
\mathbf{J}_{Lj}^{(i)} &= \begin{cases} \mathbf{b}_{j-1} & \text{for a prismatic joint} \\ \mathbf{b}_{j-1} \times \mathbf{r}_{0,ci} & \text{for a revolute joint} \end{cases} \\
\mathbf{J}_{Aj}^{(i)} &= \begin{cases} \mathbf{0} & \text{for a prismatic joint} \\ \mathbf{b}_{j-1} & \text{for a revolute joint} \end{cases}
\end{aligned}
\qquad (5\text{–}26)
$$

where $\mathbf{r}_{0,ci}$ is the position vector of the centroid of link i referred to the base coordinate frame, and $\mathbf{b}_{j-1}$ is the 3×1 unit vector along joint axis $j-1$.

Substituting expressions (5-24) into equations (5-22) and (5-23) yields

$$T = \frac{1}{2}\sum_{i=1}^{n}\left(m_i \dot{\mathbf{q}}^T \mathbf{J}_L^{(i)T} \mathbf{J}_L^{(i)} \dot{\mathbf{q}} + \dot{\mathbf{q}}^T \mathbf{J}_A^{(i)T} \mathbf{I}_i \mathbf{J}_A^{(i)} \dot{\mathbf{q}}\right) = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{H} \dot{\mathbf{q}} \tag{5-27}$$

where $\mathbf{H}$ is the $n \times n$ matrix given by

$$\mathbf{H} = \sum_{i=1}^{n}\left(m_i \mathbf{J}_L^{(i)T} \mathbf{J}_L^{(i)} + \mathbf{J}_A^{(i)T} \mathbf{I}_i \mathbf{J}_A^{(i)}\right) \tag{5-28}$$

The matrix $\mathbf{H}$ incorporates all the mass properties of the whole arm linkage, as reflected to the joint axes, and is referred to as the *manipulator inertia tensor*[1]. Note the difference between the manipulator inertia tensor and the 3×3 inertia tensors of the individual arm links. The former is a composite inertia tensor including the latter as components. The manipulator inertia tensor, however, has properties similar to those of individual inertia tensors. As shown in equation (5-28), the manipulator inertia tensor is a symmetric matrix, as is the individual inertia tensor defined by equation (5-2). The quadratic form associated with the manipulator inertia tensor represents kinetic energy, so does the individual inertia tensor. Kinetic energy is always strictly positive unless the system is at rest. The manipulator inertia tensor of equation (5-28) is positive definite, so are the individual inertia tensors. Note, however, that the manipulator inertia tensor involves Jacobian matrices, which vary with arm configuration. Therefore the manipulator inertia tensor is *configuration-dependent* and represents the instantaneous composite mass properties of the whole arm linkage at the current arm configuration.

Let H_{ij} be the $[i,j]$ component of the manipulator inertia tensor $\mathbf{H}$, then we can rewrite equation (5-27) in a scalar form so that

$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} H_{ij}\dot{q}_i\dot{q}_j \tag{5-29}$$

[1]This standard terminology is an abbreviation of *manipulator inertia tensor matrix* : strictly speaking, $\mathbf{H}$ is a matrix based on the individual inertia tensors.

Note that H_{ij} is a function of $q_1, \cdots, q_n$.

5.2.3. Deriving Lagrange's Equations of Motion

In addition to the computation of the kinetic energy we need to find the potential energy U and generalized forces in order to derive Lagrange's equations of motion. Let $\mathbf{g}$ be the 3×1 vector representing the acceleration of gravity with reference to the base coordinate frame, which is an inertial reference frame. Then the potential energy stored in the whole arm linkage is given by

$$U = \sum_{i=1}^{n} m_i \mathbf{g}^T \mathbf{r}_{0,ci} \tag{5–30}$$

where the position vector of the centroid C_i is dependent on the arm configuration. Thus the potential function is a function of $q_1, \cdots, q_n$.

Generalized forces account for all the forces and moments acting on the arm linkage except gravity forces and inertial forces. We consider the situation where actuators exert joint torques $\boldsymbol{\tau} = [\tau_1, \cdots, \tau_n]^T$ at individual joints and an external force and moment $\mathbf{F}_{ext}$ is applied at the arm's endpoint while in contact with the environment. Generalized forces can be obtained by computing the virtual work done by these forces. In equation (4-9), let us replace the endpoint force exerted by the manipulator by the negative external force $-\mathbf{F}_{ext}$. Then the virtual work is given by

$$\delta\text{Work} = \boldsymbol{\tau}^T \delta\mathbf{q} + \mathbf{F}_{ext}^T \delta\mathbf{p} = (\boldsymbol{\tau} + \mathbf{J}^T \mathbf{F}_{ext})^T \delta\mathbf{q} \tag{5–31}$$

By comparing this expression with the one in terms of generalized forces $\mathbf{Q} = [Q_1, \cdots, Q_n]^T$, given by

$$\delta\text{Work} = \mathbf{Q}^T \delta\mathbf{q} \tag{5–32}$$

we can identify the generalized forces as

$$\mathbf{Q} = \boldsymbol{\tau} + \mathbf{J}^T\mathbf{F}_{ext} \tag{5-33}$$

Using the total kinetic energy (5-29) and the total potential energy (5-30), we can now derive Lagrange's equations of motion. From equation (5-29), the first term in equation (5-21) is computed as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \frac{d}{dt}\left(\sum_{j=1}^{n} H_{ij}\dot{q}_j\right) = \sum_{j=1}^{n} H_{ij}\ddot{q}_j + \sum_{j=1}^{n} \frac{dH_{ij}}{dt}\dot{q}_j \tag{5-34}$$

Note that H_{ij} is a function of $q_1, \cdots, q_n$, so that the time derivative of H_{ij} is given by

$$\frac{dH_{ij}}{dt} = \sum_{k=1}^{n} \frac{\partial H_{ij}}{\partial q_k}\frac{dq_k}{dt} = \sum_{k=1}^{n} \frac{\partial H_{ij}}{\partial q_k}\dot{q}_k \tag{5-35}$$

The second term in equation (5-21) includes the partial derivative of the kinetic energy, given by

$$\frac{\partial T}{\partial q_i} = \frac{\partial}{\partial q_i}\left(\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} H_{jk}\dot{q}_j\dot{q}_k\right) = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial H_{jk}}{\partial q_i}\dot{q}_j\dot{q}_k \tag{5-36}$$

since H_{jk} depends on q_i. The gravity term G_i is obtained by taking the partial derivative of the potential energy:

$$G_i = \frac{\partial U}{\partial q_i} = \sum_{j=1}^{n} m_j \mathbf{g}^T \frac{\partial \mathbf{r}_{0,cj}}{\partial q_i} = \sum_{j=1}^{n} m_j \mathbf{g}^T \mathbf{J}_{Li}^{(j)} \tag{5-37}$$

since the partial derivative of the position vector $\mathbf{r}_{0,cj}$ with respect to q_i is the same as the i-th column vector of the Jacobian matrix $\mathbf{J}_{Li}^{(j)}$ defined by equations (5-24)-(5-26). Substituting expressions (5-34) through (5-37) into (5-21) yields

$$\sum_{j=1}^{n} H_{ij}\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} h_{ijk}\dot{q}_j\dot{q}_k + G_i = Q_i \qquad i = 1, \cdots, n \tag{5-38}$$

where

$$h_{ijk} = \frac{\partial H_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial H_{jk}}{\partial q_i} \tag{5-39}$$

and

$$G_i = \sum_{j=1}^{n} m_j \mathbf{g}^T \mathbf{J}_{L\,i}^{(j)} \tag{5-40}$$

The first term represents inertia torques, including interaction torques, while the second term accounts for the Coriolis and centrifugal effects, and the last term is the gravity torque. It is important to note that interactive inertia torques $H_{ij}\ddot{q}_j$ $(j \neq i)$ result from the off-diagonal elements of the manipulator inertia tensor and that the Coriolis and centrifugal torques $h_{ijk}\dot{q}_j\dot{q}_k$ arise because the manipulator inertia tensor is configuration dependent. Equation (5-38) is the same as equation (5-13) derived from Newton-Euler equations. Thus the Lagrangian formulation provides the closed-form dynamic equations directly.

Example 5-2

Let us derive closed-form dynamic equations for the two degree-of-freedom planar manipulator shown in Figure 5-2, using Lagrange's equations of motion.

We begin by computing the manipulator inertia tensor **H**. From equation (5-10), velocities of the centroids C_1 and C_2 can be written as

$$\mathbf{v}_{c1} = \begin{bmatrix} -l_{c1}\sin(\theta_1) & 0 \\ l_{c1}\cos(\theta_1) & 0 \end{bmatrix}\dot{\mathbf{q}}$$

$$\mathbf{v}_{c2} = \begin{bmatrix} -l_1\sin(\theta_1) - l_{c2}\sin(\theta_1+\theta_2) & -l_{c2}\sin(\theta_1+\theta_2) \\ l_1\cos(\theta_1) + l_{c2}\cos(\theta_1+\theta_2) & l_{c2}\cos(\theta_1+\theta_2) \end{bmatrix}\dot{\mathbf{q}} \tag{5-41}$$

The above 2×2 matrices are the Jacobian matrices $\mathbf{J}_L^{(i)}$ of equation (5-24). The angular velocities are associated with the Jacobian matrices $\mathbf{J}_A^{(i)}$, which are 1×2 row-vectors in this planar case:

$$\begin{aligned} \omega_1 &= \dot{\theta}_1 = [1 \quad 0]\dot{\mathbf{q}} \\ \omega_2 &= \dot{\theta}_1 + \dot{\theta}_2 = [1 \quad 1]\dot{\mathbf{q}}, \end{aligned} \tag{5-42}$$

Substituting the above expressions into equation (5-28), we obtain the manipulator inertia tensor

$$\mathbf{H} = \begin{bmatrix} m_1 l_{c1}^2 + I_1 + m_2(l_1^2 + l_{c2}^2 + 2\, l_1 l_{c2} \cos\theta_2) + I_2 & m_2\, l_1 l_{c2} \cos\theta_2 + m_2\, l_{c2}^2 + I_2 \\ m_2\, l_1 l_{c2} \cos\theta_2 + m_2\, l_{c2}^2 + I_2 & m_2\, l_{c2}^2 + I_2 \end{bmatrix} \tag{5-43}$$

The components of the above inertia tensor are the coefficients of the first term of equation (5-38). The second term is determined by substituting equation (5-43) into equation (5-39).

$$\left[\begin{aligned} & h_{111} = 0 \ , \ h_{122} = -m_2\, l_1 l_{c2} \sin\theta_2 \ , \ h_{112} + h_{121} = -2\, m_2\, l_1 l_{c2} \sin\theta_2 \\ & h_{211} = m_2\, l_1 l_{c2} \sin\theta_2 \ , \ h_{222} = 0 \ , \ h_{212} + h_{221} = 0 \end{aligned}\right. \tag{5-44}$$

The third term in Lagrange's equations of motion, i.e., the gravity term, is derived from equation (5-40) using the Jacobian matrices in equation (5-41) :

$$\begin{aligned} G_1 &= \mathbf{g}^T [m_1 \mathbf{J}_{L1}^{(1)} + m_2 \mathbf{J}_{L1}^{(2)}] \\ G_2 &= \mathbf{g}^T [m_1 \mathbf{J}_{L2}^{(1)} + m_2 \mathbf{J}_{L2}^{(2)}] \end{aligned} \tag{5-45}$$

Substituting equations (5-43), (5-44) and (5-45) into equation (5-38) yields

$$\begin{aligned} & H_{11}\ddot{\theta}_1 + H_{12}\ddot{\theta}_2 + h_{122}\dot{\theta}_2^2 + (h_{112}+h_{121})\dot{\theta}_1\dot{\theta}_2 + G_1 = \tau_1 \\ & H_{22}\ddot{\theta}_2 + H_{12}\ddot{\theta}_1 + h_{211}\dot{\theta}_1^2 + G_2 = \tau_2 \end{aligned} \tag{5-46}$$