Planar Elbow Manipulator with Remotely Driven Link

Now we illustrate the use of Lagrangian equations in a situation where the generalized coordinates are not the joint variables defined in earlier chapters. Consider again the planar elbow manipulator, but suppose now that both joints are driven by motors mounted at the base. The first joint is turned directly by one of the motors, while the other is turned via a gearing mechanism or a timing belt (see Figure 9.8). In this case one should choose

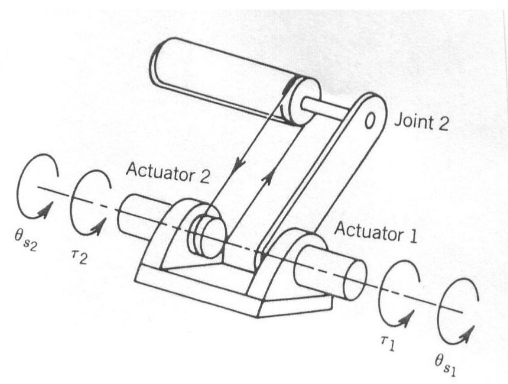

Figure 9.8: Two-link revolute joint arm with remotely driven link.

the generalized coordinates as shown in Figure 9.9, because the angle p_2 is determined by

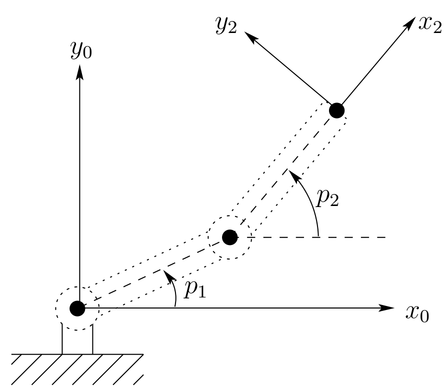

Figure 9.9: Generalized coordinates for robot of Figure 6.4.

driving motor number 2, and is not affected by the angle p_1. We will derive the dynamical equations for this configuration, and show that some simplifications will result.

Since p_1 and p_2 are not the joint angles used earlier, we cannot use the velocity Jacobians derived in Chapter 5 in order to find the kinetic energy of each link. Instead, we have to carry out the analysis directly. It is easy to see that

$$\boldsymbol{v}_{c1} = \begin{bmatrix} -\ell_{c1}\sin p_1 \\ \ell_{c1}\cos p_1 \\ 0 \end{bmatrix} \dot{p}_1, \quad \boldsymbol{v}_{c2} = \begin{bmatrix} \ell_1 \sin p_1 & -\ell_{c2}\sin p_2 \\ \ell_1\cos p_1 & \ell_{c2}\cos p_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} \tag{9.89}$$

$$\boldsymbol{\omega}_1 = \dot{p}_1 \boldsymbol{k}, \qquad \boldsymbol{\omega}_2 = \dot{p}_2 \boldsymbol{k}. \tag{9.90}$$

Hence the kinetic energy of the manipulator equals

$$K = \frac{1}{2}\dot{\boldsymbol{p}}^T D(\boldsymbol{p})\dot{\boldsymbol{p}} \tag{9.91}$$

where

$$D(\boldsymbol{p}) = \begin{bmatrix} m_1\ell_{c1}^2 + m_2\ell_1^2 + I_1 & m_2\ell_1\ell_{c2}\cos(p_2 - p_1) \\ m_2\ell_1\ell_{c2}\cos(p_2 - p_1) & m_2\ell_{c2}^2 + I_2. \end{bmatrix} \tag{9.92}$$

Computing the Christoffel symbols as in (9.58) gives

$$\begin{aligned} c_{111} &= \frac{1}{2}\frac{\partial d_{11}}{\partial p_1} = 0 \\ c_{121} &= c_{211} = \frac{1}{2}\frac{\partial d_{11}}{\partial p_2} = 0 \\ c_{221} &= \frac{\partial d_{12}}{\partial p_2} - \frac{1}{2}\frac{\partial d_{22}}{\partial p_1} = -m_2\ell_1\ell_{c2}\sin(p_2 - p_1) \\ c_{112} &= \frac{\partial d_{21}}{\partial p_1} - \frac{1}{2}\frac{\partial d_{11}}{\partial p_2} = m_2\ell_1\ell_{c2}\sin(p_2 - p_1) \\ c_{212} &= = c_{122} = \frac{1}{2}\frac{\partial d_{22}}{\partial p_1} = 0 \\ c_{222} &= \frac{1}{2}\frac{\partial d_{22}}{\partial p_2} = 0. \end{aligned} \tag{9.93}$$

Next, the potential energy of the manipulator, in terms of p_1 and p_2, equals

$$P = m_1 g\ell_{c1}\sin p_1 + m_2 g(\ell_1\sin p_1 + \ell_{c2}\sin p_2). \tag{9.94}$$

Hence the gravitational generalized forces are

$$\begin{aligned} \phi_1 &= (m_1\ell_{c1} + m_2\ell_1)g\cos p_1 \\ \phi_2 &= m_2\ell_{c2}g\cos p_2. \end{aligned}$$

Finally, the equations of motion are

$$\begin{aligned} d_{11}\ddot{p}_1 + d_{12}\ddot{p}_2 + c_{221}\dot{p}_2^2 + \phi_1 &= \tau_1 \\ d_{21}\ddot{p}_1 + d_{22}\ddot{p}_2 + c_{112}\dot{p}_1^2 + \phi_2 &= \tau_2. \end{aligned} \tag{9.95}$$

Comparing (9.95) and (9.87), we see that by driving the second joint remotely from the base we have eliminated the Coriolis forces, but we still have the centrifugal forces coupling the two joints.