Trajectory Generation
& Motion Planning

METR 4202: Advanced Control & Robotics

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Lecture # 5 August 20, 2012

(with Motion Planning Figures from Latombe, CS326A)

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RoboticsCourseWare Contributor

Schedule

Week	Date	Lecture (M: 12-1:30, 43-102)
1	23-Jul	Introduction
2	30-Jul	Representing Position & Orientation & State (Frames, Transformation Matrices & Affine Transformations)
3	6-Aug	Robot Kinematics and Dynamics
4	13-Aug	Robot Dynamics & Control
5	20-Aug	[Trajectory Generation+Control] Obstacle Avoidance & Motion Planning
6	27-Aug	Sensors, Measurement and Perception
7	3-Sep	Localization and Navigation
8	10-Sep	State-space modelling & Controller Design
9	17-Sep	Vision-based control
	24-Sep	*Study break*
10	1-Oct	*Public Holiday*
11	8-Oct	Robot Machine Learning
12	15-Oct	Guest Lecture
13	22-Oct	Wrap-up & Course Review

Revisiting The Jacobian

- I told you (Lec 4, Slide 19):

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_i \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial \theta_1} & \frac{\partial x_1}{\partial \theta_2} & \cdots & \frac{\partial x_1}{\partial \theta_j} \\ \frac{\partial x_2}{\partial \theta_1} & \frac{\partial x_2}{\partial \theta_2} & \cdots & \frac{\partial x_2}{\partial \theta_j} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_i}{\partial \theta_1} & \frac{\partial x_i}{\partial \theta_2} & \cdots & \frac{\partial x_i}{\partial \theta_j} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_j \end{bmatrix}$$

- True, but we can be more "explicit"

Jacobian: **Explicit Form**

- For a serial chain (robot): The velocity of a link with respect to the proceeding link is dependent on the type of link that connects them
- If the joint is **prismatic** (**ϵ=1**), then $\mathbf{v}_i = \frac{dz}{dt}$
- If the joint is **revolute** (**ϵ=0**), then $\omega = \frac{d\theta}{dt}$ (in the $\hat{k}$ direction)

$$\therefore v = \sum_{i=1}^{N} \left(\varepsilon_i v_i + \bar{\varepsilon}_i \left(\omega_i \times \mathbf{p}_{i-1}^{i} \right) \right) \qquad \omega = \sum_{i=1}^{N} \left(\bar{\varepsilon}_i \left(\dot{\boldsymbol{\theta}}_i \right) \right) = \sum_{i=1}^{N} \left(\bar{\varepsilon}_i \mathbf{z}_i \left(\dot{\theta}_i \right) \right)$$

$$\rightarrow v = J_v \dot{\mathbf{q}} \qquad \boldsymbol{\omega} = J_\omega \dot{\mathbf{q}}$$

- Combining them (with **v**=(Δx, Δθ))

$$J = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix}$$

Jacobian: **Explicit Form [2]**

- The overall Jacobian takes the form

$$J = \begin{bmatrix} \dfrac{\partial x_P}{\partial q_1} & \cdots & \dfrac{\partial x_P}{\partial q_n} \\ \bar{\varepsilon}_1 z_1 & \cdots & \bar{\varepsilon}_1 z_n \end{bmatrix}$$

- The Jacobian for a particular frame (F) can be expressed:

$${}^F J = \begin{bmatrix} \dfrac{\partial\, {}^F x_P}{\partial q_1} & \cdots & \dfrac{\partial\, {}^F x_P}{\partial q_n} \\ \bar{\varepsilon}_1\, {}^F z_1 & \cdots & \bar{\varepsilon}_1\, {}^F z_n \end{bmatrix}$$

Where: ${}^F\mathbf{z}_i = {}^F_i R\, {}^i\mathbf{z}_i$ & ${}^i\mathbf{z}_i = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$

Correction! Dynamics – Langrangian Mechanics

- Alternatively, we can use Langrangian Mechanics to compute the dynamics of a manipulator (or other robotic system)
- The Langrangian is defined as the difference between the Kinetic and Potential energy in the system
- Using this formulation and the concept of virtual work we can find the forces and torques acting on the system.
- This may seem more involved but is often easier to formulate for complex systems

$$L = K - P$$

$$\mathbf{F} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{x}}}\right) - \frac{\partial L}{\partial \mathbf{x}}$$

$$\tau = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta}$$

Dynamics – Langrangian Mechanics [2]

$L = K - P, \dot{\boldsymbol{\theta}}$: Generalized Velocities, M : Mass Matrix

$$\boldsymbol{\tau} = \sum_{i=1}^{N} \tau_i = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\boldsymbol{\theta}}}\right) - \frac{\partial K}{\partial \boldsymbol{\theta}} + \frac{\partial P}{\partial \boldsymbol{\theta}}$$

$$K = \tfrac{1}{2}\dot{\theta}^T M(\theta)\,\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{\theta}}\left(\tfrac{1}{2}\dot{\theta}^T M(\theta)\,\dot{\theta}\right)\right) = \frac{d}{dt}\left(M\dot{\theta}\right) = M\ddot{\theta} + \dot{M}\dot{\theta}$$

$$\rightarrow \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\theta}}\right) - \frac{\partial K}{\partial \theta} = \left[M\ddot{\theta} + \dot{M}\dot{\theta}\right] - \left[\tfrac{1}{2}\dot{\theta}^T M(\theta)\,\dot{\theta}\right] = M\ddot{\theta} + \underbrace{\left\{\dot{M}\dot{\theta} - \tfrac{1}{2}\begin{bmatrix}\dot{\theta}^T \frac{\partial M}{\partial \theta_1}\dot{\theta} \\ \vdots \\ \dot{\theta}^T \frac{\partial M}{\partial \theta_n}\dot{\theta}\end{bmatrix}\right\}}_{\mathbf{v}(\theta,\dot{\theta})}$$

$$\mathbf{v}\left(\theta,\dot{\theta}\right) = \underbrace{C(\theta)\left[\dot{\theta}^2\right]}_{\text{Centrifugal}} + \underbrace{B(\theta)\left[\dot{\theta}\dot{\theta}\right]}_{\text{Coriolis}}$$

$$\Rightarrow \boldsymbol{\tau} = M(\theta)\ddot{\boldsymbol{\theta}} + \mathbf{v}\left(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}\right) + \mathbf{g}\left(\boldsymbol{\theta}\right)$$

Dynamics – Langrangian Mechanics [3]

- The Mass Matrix: Determining via the Jacobian!

$$K = \sum_{i=1}^{N} K_i$$

$$K_i = \tfrac{1}{2}\left(m_i v_{C_i}^T v_{C_i} + \omega_i^T I_{C_i}\omega_i\right)$$

$$v_{C_i} = J_{v_i}\dot{\theta} \qquad J_{v_i} = \begin{bmatrix} \frac{\partial \mathbf{p}_{C_1}}{\partial \theta_1} & \cdots & \frac{\partial \mathbf{p}_{C_i}}{\partial \theta_i} & \underbrace{0}_{i+1} & \cdots & \underbrace{0}_{n} \end{bmatrix}$$

$$\omega_i = J_{\omega_i}\dot{\theta} \qquad J_{\omega_i} = \begin{bmatrix} \bar{\varepsilon}_1 Z_1 & \cdots & \bar{\varepsilon}_i Z_i & \underbrace{0}_{i+1} & \cdots & \underbrace{0}_{n} \end{bmatrix}$$

$$\therefore M = \sum_{i=1}^{N}\left(m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T I_{C_i} J_{\omega_i}\right)$$

! M is symmetric, positive definite $\therefore m_{ij} = m_{ji}, \dot{\boldsymbol{\theta}}^T M \dot{\boldsymbol{\theta}} > 0$

Generalized Coordinates

- A significant feature of the Lagrangian Formulation is that any convenient coordinates can be used to derive the system.
- Go from Joint → Generalized
  - Define **p**: $d\mathbf{p} = \mathbf{J}d\mathbf{q}$

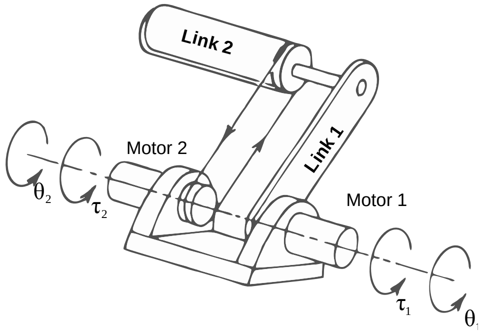
$$\mathbf{q} = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} \rightarrow \mathbf{p} = \begin{bmatrix} p_1 & \cdots & p_n \end{bmatrix}$$

➔Thus: the kinetic energy and gravity terms become

$$KE = \tfrac{1}{2}\dot{\mathbf{p}}^T \mathbf{H}^* \dot{\mathbf{p}} \qquad \mathbf{G}^* = \left(\mathbf{J}^{-1}\right)^T \mathbf{G}$$

where: $\mathbf{H}^* = \left(\mathbf{J}^{-1}\right)^T \mathbf{H}\mathbf{J}^{-1}$

Motivating Example: Remotely Driven 2DOF Manipulator



- Introduce $Q_1=\tau_1+\tau_2$ and $Q_2=\tau_2$
- Derivation posted to website

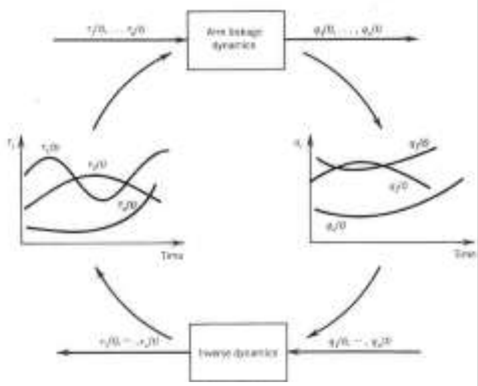
Graphic based on Asada & Slotine p. 112

Inverse Dynamics

- Forward dynamics governs the dynamic responses of a manipulator arm to the input torques generated by the actuators.
- The inverse problem:
  - Going from joint angles to torques
  - Inputs are desired trajectories described as functions of time

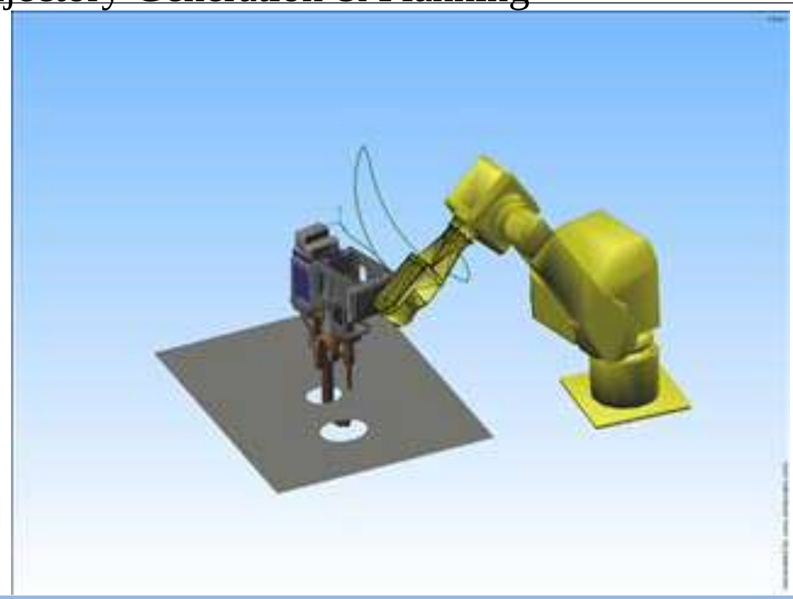
    $\mathbf{q} = [q_1 \quad \dots \quad q_n] \rightarrow [\theta_1(t) \quad \theta_2(t) \quad \theta_3(t)]$
  - Outputs are joint torques to be applied at each instance

    $\boldsymbol{\tau} = [\tau_1 \quad \dots \quad \tau_n]$
- Computation "big" (6DOF arm: 66,271 multiplications), but not scary (4.5 ms on PDP11/45)



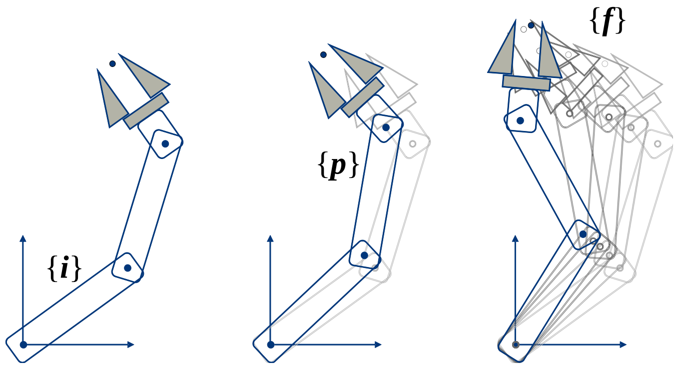
Graphic from Asada & Slotine p. 119

Trajectory Generation & Planning



Trajectory Generation

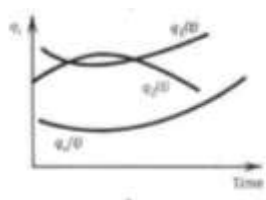
- The goal is to get from an initial position $\{\boldsymbol{i}\}$ to a final position $\{\boldsymbol{f}\}$ via a path points $\{\boldsymbol{p}\}$



Joint Space

Consider only the **joint positions** as a function of time

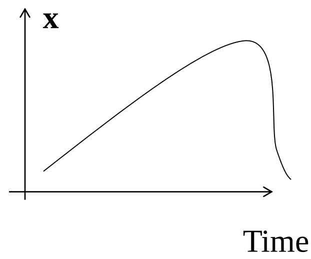
- + Since we control the joints, this is more direct
- -- If we want to follow a particular trajectory, <u>not easy</u>
  - at best lots of intermediate points
  - No guarantee that you can solve the Inverse Kinematics for all path points



Cartesian Workspace

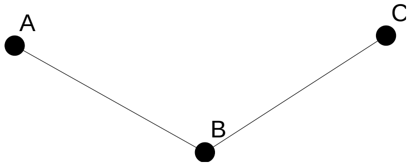
Consider the **Cartesian positions** as a function of time

- + Can track shapes exactly
- -- We need to solve the inverse kinematics and dynamics



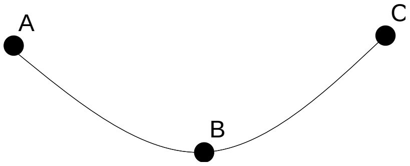
Polynomial Trajectories

- Straight line Trajectories



- Simpler

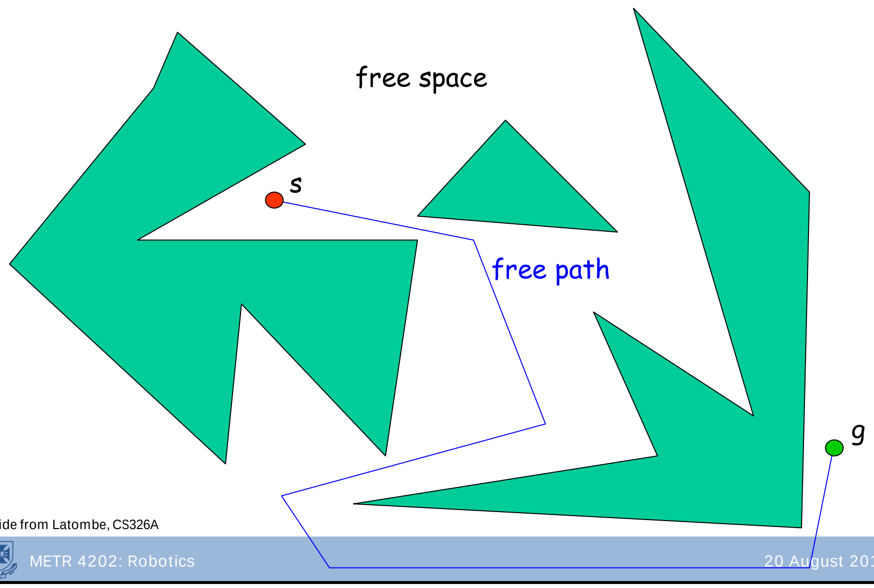
- Polynomial Trajectories



$$u(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

- Parabolic blends are smoother
- Use "pseudo via points"

Problem



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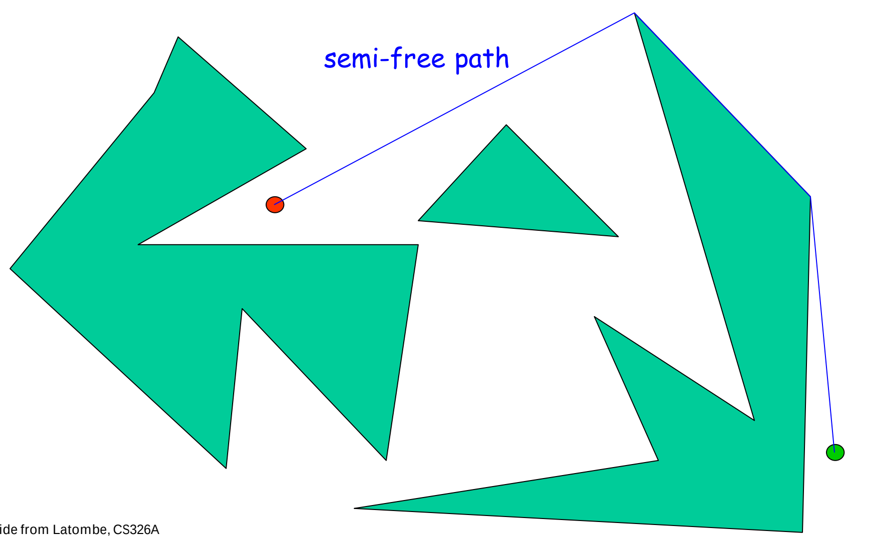
Problem



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Motion-Planning Framework

Continuous representation

↓

Discretization

↓

Graph searching
(blind, best-first, A*)

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Path-Planning Approaches

- Roadmap
  Represent the connectivity of the free space by a network of 1-D curves
- Cell decomposition
  Decompose the free space into simple cells and represent the connectivity of the free space by the adjacency graph of these cells
- Potential field
  Define a function over the free space that has a global minimum at the goal configuration and follow its steepest descent

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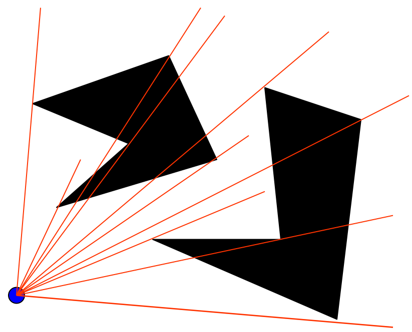
I. Rotational Sweep



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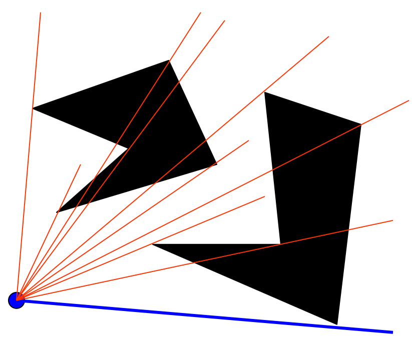
Rotational Sweep



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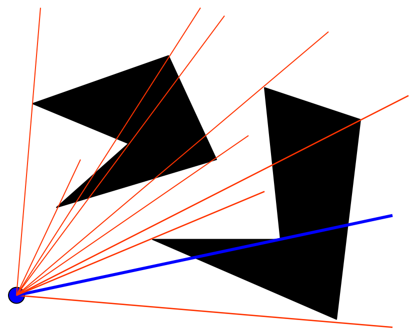

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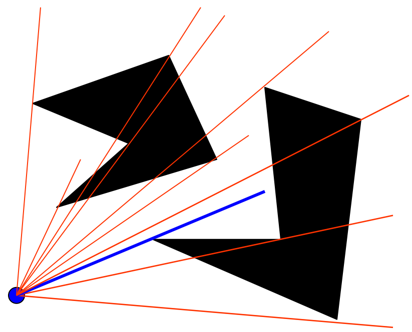

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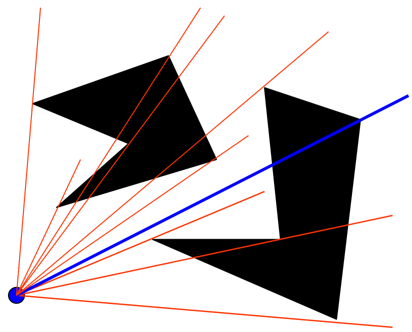
Rotational Sweep



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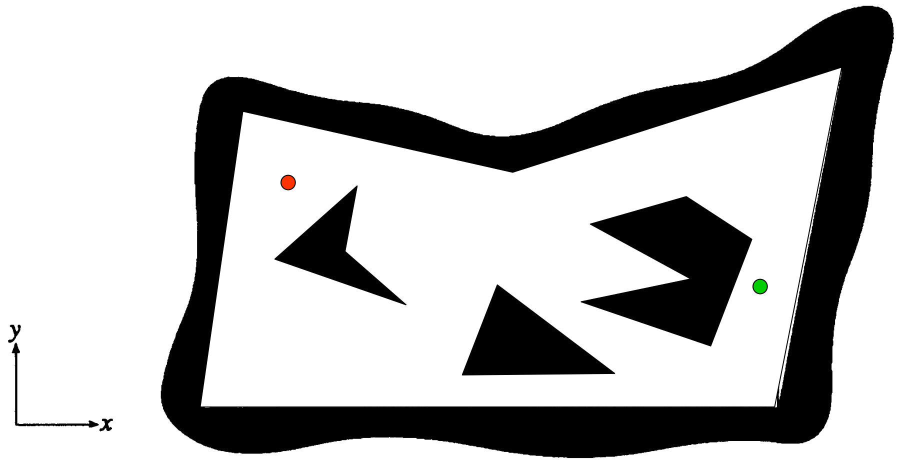
II. Cell-Decomposition Methods

Two classes of methods:

- Exact cell decomposition
  - The free space **F** is represented by a collection of non-overlapping cells whose union is exactly **F**
  - Example: trapezoidal decomposition
- Approximate cell decomposition
  - **F** is represented by a collection of non-overlapping cells whose union is contained in **F**
    Examples: quadtree, octree, 2n-tree

Trapezoidal decomposition



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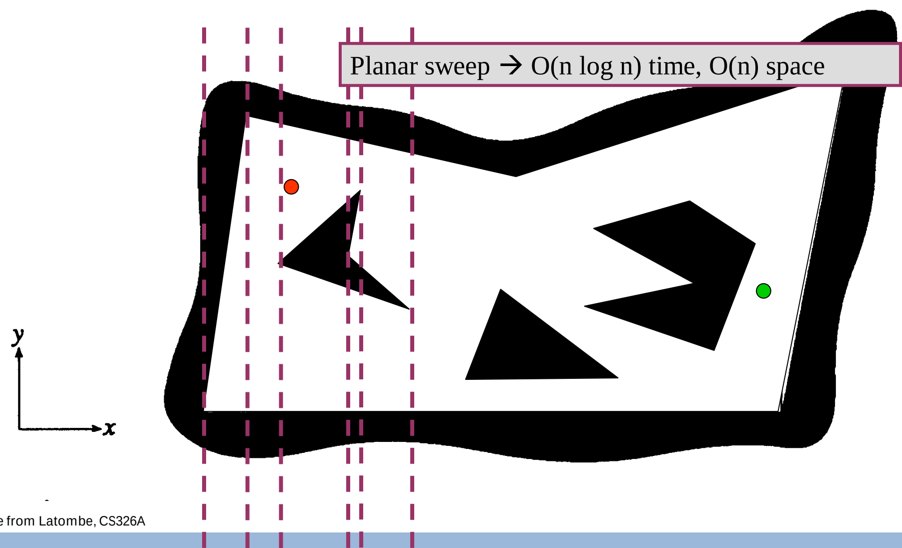
Trapezoidal decomposition



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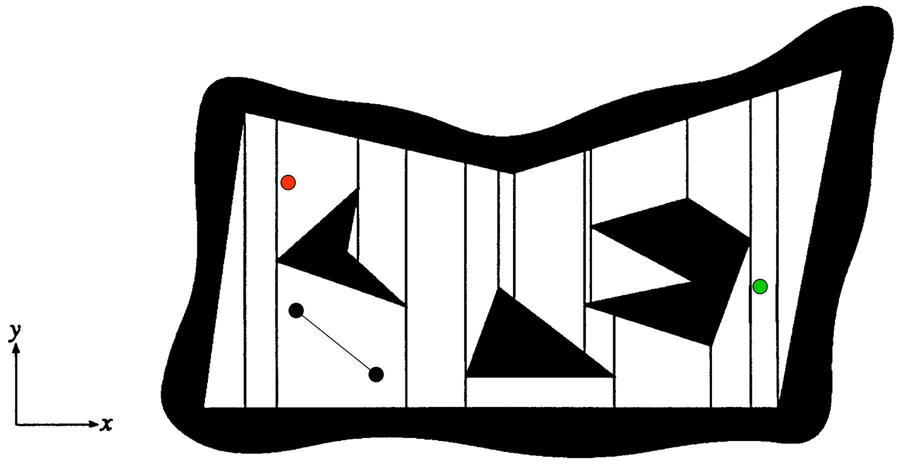
Trapezoidal decomposition



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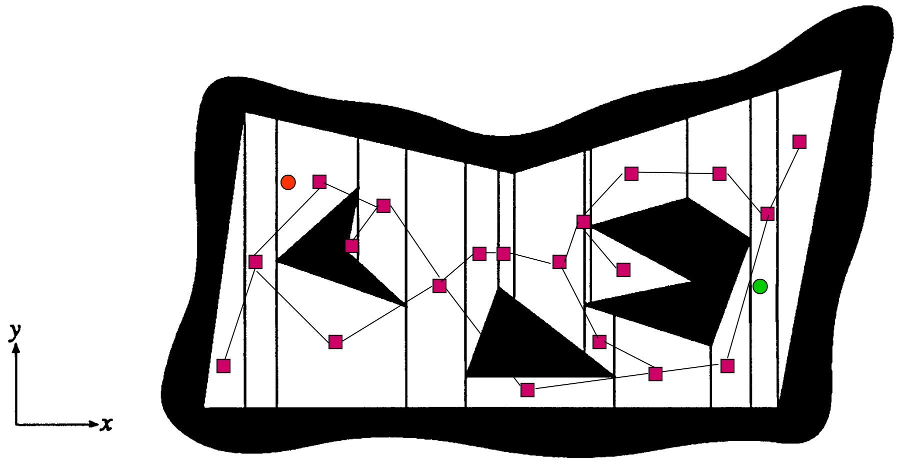
Trapezoidal decomposition



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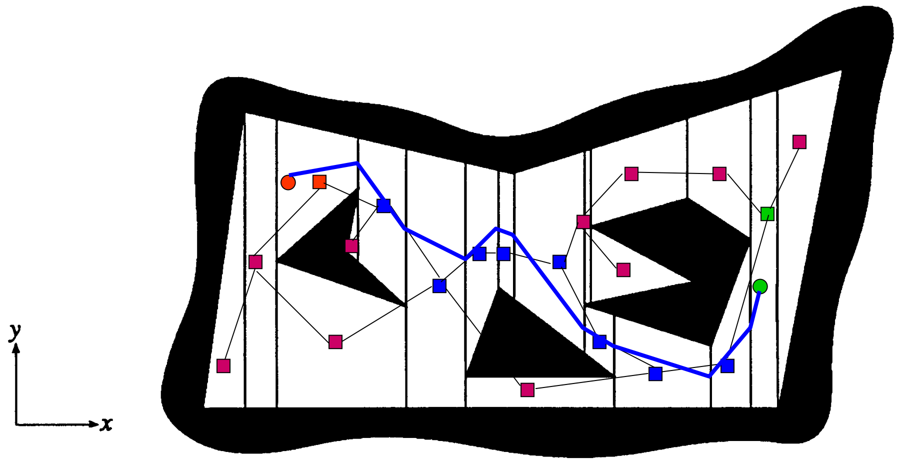
Trapezoidal decomposition



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III. Roadmap Methods

- **Visibility graph**
- **Voronoi diagram**
- Silhouette
  First complete general method that applies to spaces of any dimension and is singly exponential in # of dimensions [Canny, 87]
- **Probabilistic roadmaps (PRMS) and Rapidly-exploring Randomized Trees (RRTs)**

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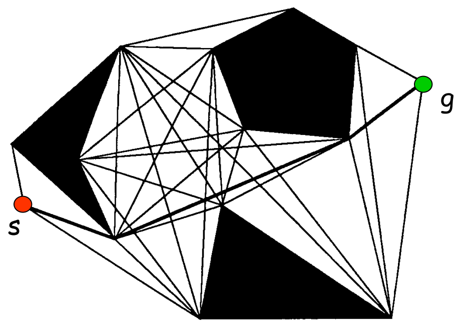
Roadmap Methods

- **Visibility graph**
  Introduced in the Shakey project at SRI in the late 60s.
  Can produce shortest paths in 2-D configuration spaces



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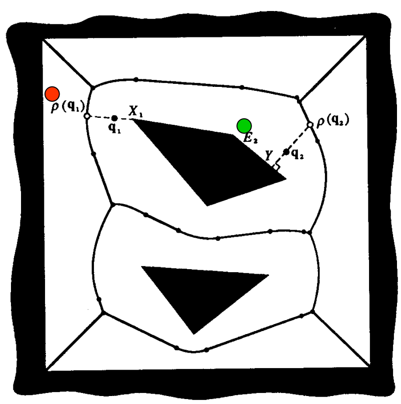
Roadmap Methods

- Voronoi diagram
  Introduced by Computational Geometry researchers. Generate paths that maximizes clearance.

  O(n log n) time
  O(n) space



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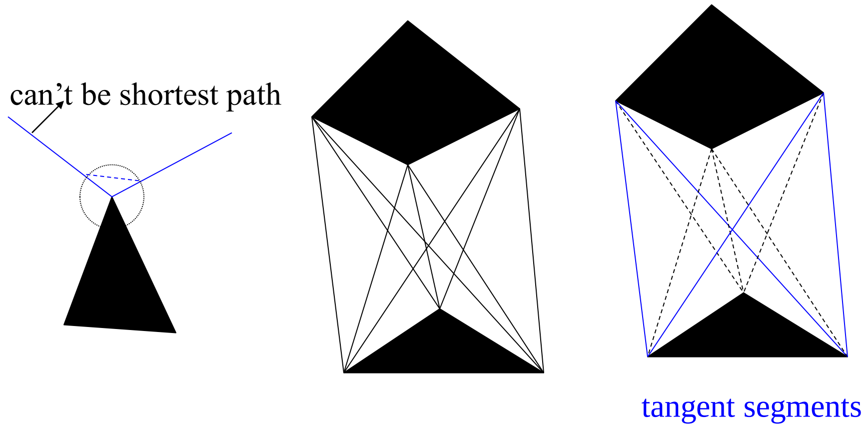
II.Visibility Graph



→ Eliminate concave obstacle vertices

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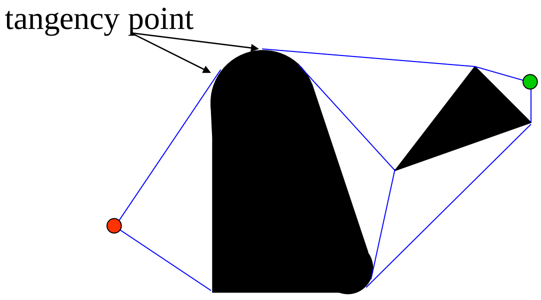
Generalized (Reduced) -- Visibility Graph



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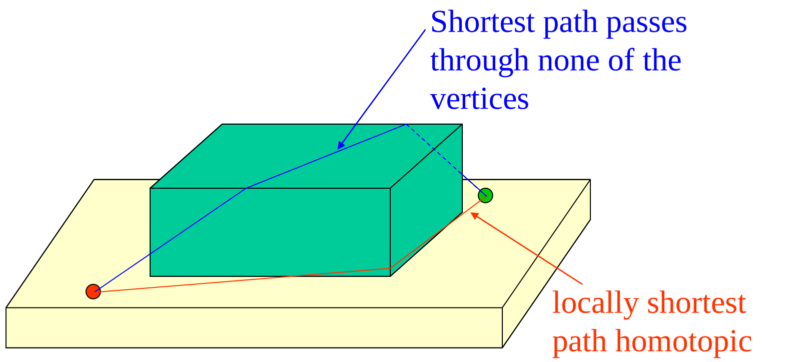
Three-Dimensional Space



Computing the shortest collision-free path in a polyhedral space is NP-hard

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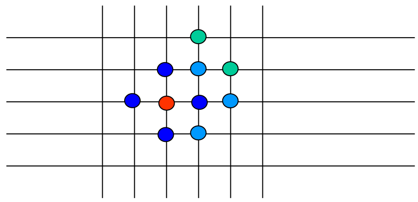
Sketch of Grid Algorithm (with best-first search)

- Place regular grid G over space
- Search G using best-first search algorithm with potential as heuristic function



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Simple Algorithm (for Visibility Graphs)

- Install all obstacles vertices in VG, plus the start and goal positions
- For every pair of nodes u, v in VG

    If segment(u,v) is an obstacle edge then

        insert (u,v) into VG

    else

    for every obstacle edge e

        if segment(u,v) intersects e

            then go up to segment

  insert (u,v) into VG
- Search VG using A*

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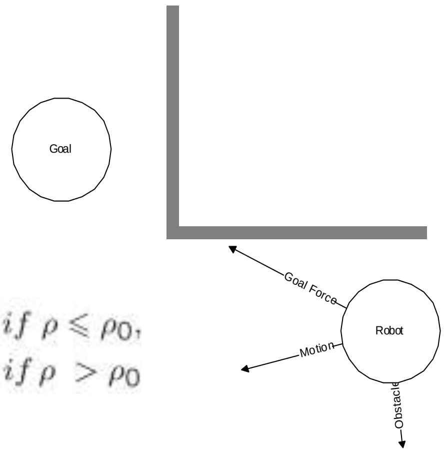
IV. Potential Field Methods

- Approach initially proposed for real-time collision avoidance [Khatib, 86]

$$F_{Goal} = -k_p(x - x_{Goal})$$

$$F_{Obstacle} = \begin{cases} \eta \left(\frac{1}{\rho} - \frac{1}{\rho_0}\right) \frac{1}{\rho^2} \frac{\partial \rho}{\partial x} & if\ \rho \leqslant \rho_0, \\ 0 & if\ \rho > \rho_0 \end{cases}$$



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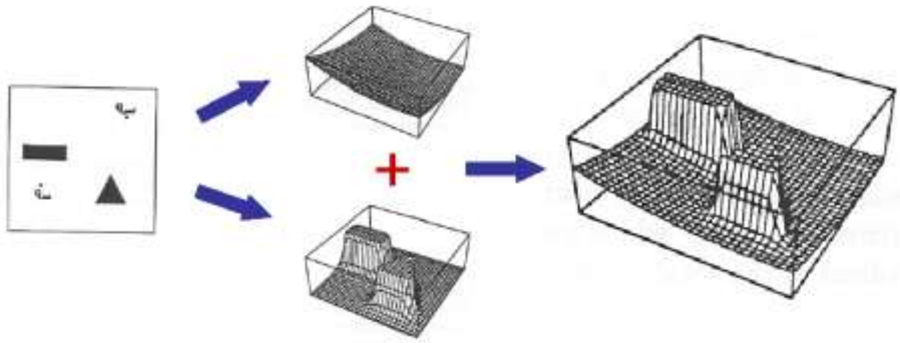
Attractive and Repulsive fields



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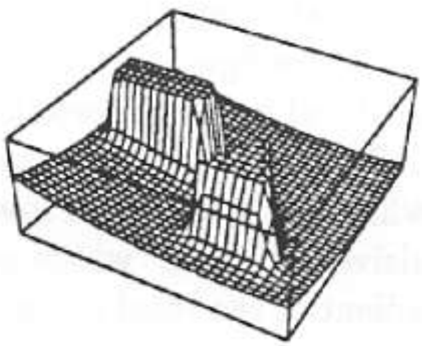
Local-Minimum Issue



- Perform best-first search (possibility of combining with approximate cell decomposition)
- Alternate descents and random walks
- Use local-minimum-free potential (navigation function)

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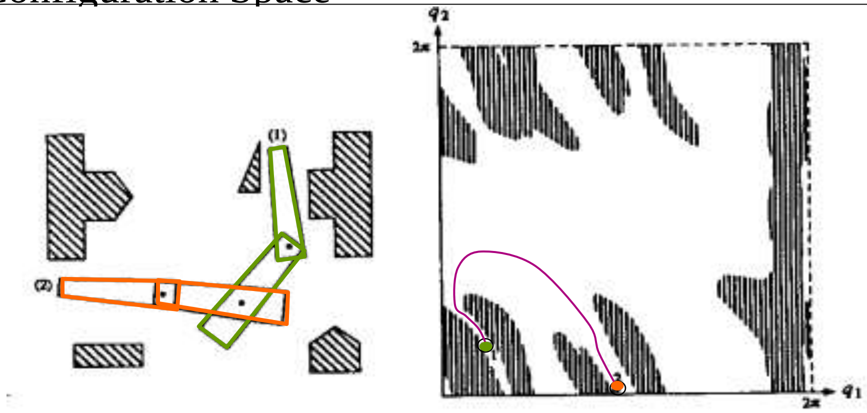
Configuration Space



- A robot configuration is a specification of the positions of all robot points relative to a fixed coordinate system
- Usually a configuration is expressed as a "vector" of position/orientation parameters

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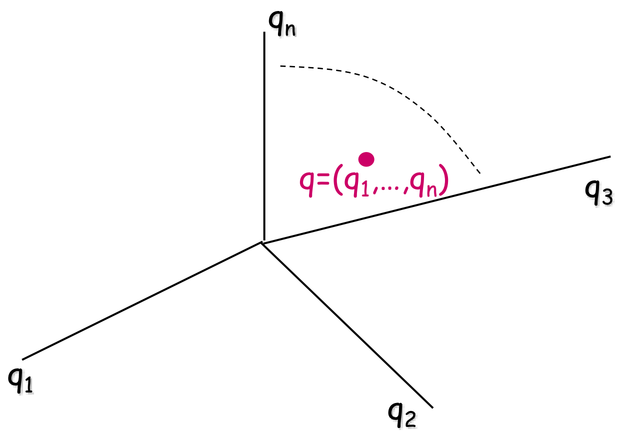
Motion Planning in C-Space



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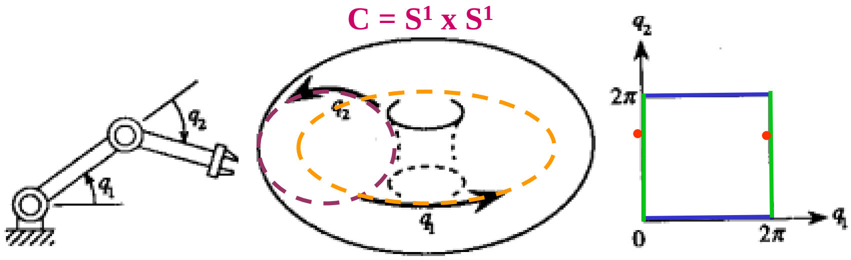
Configuration Space of a Robot

- Space of all its possible configurations
- But the topology of this space is usually not that of a Cartesian space



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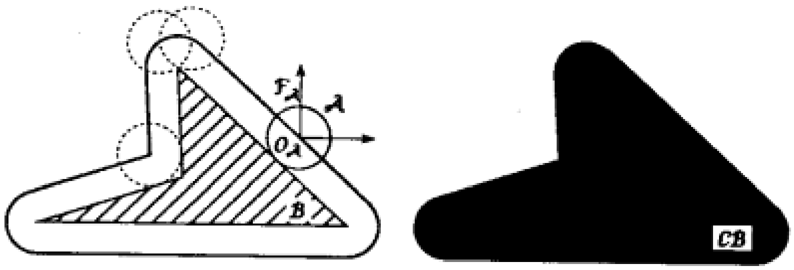
Disc Robot in 2-D Workspace



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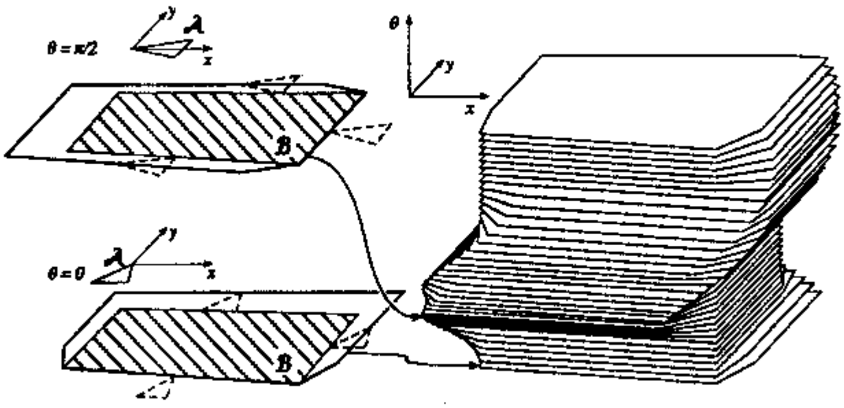
Rigid Robot Translating and Rotating in 2-D



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Cool Robotics Share