



## Position & Orientation & State

*“At home with Homogenous Transformations”*

METR 4202: Advanced Control & Robotics

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Lecture # 2

July 30, 2012

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## Continuing from last week...

- Robotics Definition:

“A robot is a reprogrammable, multifunctional manipulator designed to move material, parts, tools, or specialized devices through variable programmed motions for the performance of a variety of tasks.”

(Robotics Institute of America)

It is a machine which has some ability to interact with physical objects and to be given electronic programming to do a specific task or to do a whole range of tasks or actions.

(Wikipedia)

Programmable electro-mechanical systems that adapt to identify and leverage a **structural characteristic** of the environment

(Surya)



## Types of Robotics Systems

- Manipulators



- Multiple



- Mobile



### Enabling Mathematics:

- Computational Kinematics
- Operational Space

- Behaviour based “Reflexive” control rules

- Probabilistic methods

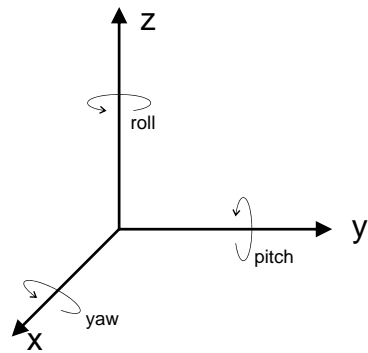


## Schedule

Week	Date	Lecture (M: 12-1:30, 43-102)
1	23-Jul	Introduction
2	30-Jul	<b>Representing Position &amp; Orientation &amp; State (Frames, Transformation Matrices &amp; Affine Transformations)</b>
3	6-Aug	Robot Kinematics and Dynamics
4	13-Aug	Robot Dynamics & Control
5	20-Aug	Obstacle Avoidance & Motion Planning
6	27-Aug	Sensors, Measurement and Perception
7	3-Sep	Localization and Navigation
8	10-Sep	State-space modelling & Controller Design
9	17-Sep	Vision-based control
	24-Sep	<i>Study break</i>
10	1-Oct	Uncertainty/POMDPs
11	8-Oct	Robot Machine Learning
12	15-Oct	Guest Lecture
13	22-Oct	Wrap-up & Course Review

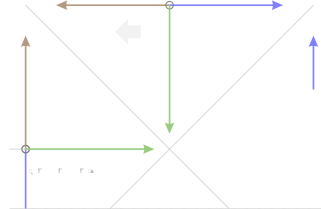


## Today's Lecture is about: Frames & Their Mathematics

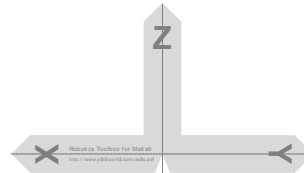


- Make one (online):

- SpnS Template



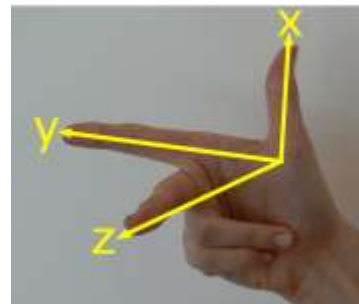
- Peter Corke's template



## Don't Confuse a Frame with a Point

- Points
  - Position Only –  
Doesn't Encode Orientation

- Frame
  - Encodes both position  
and orientation
  - Has a "handedness"



## Kinematics Definition

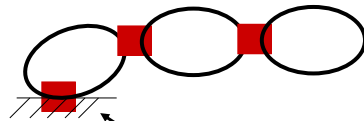
- **Kinematics**: The study of motion in space (without regard to the forces which cause it)

- **Assume:**

Points with *right-hand Frames*

*Rigid-bodies* in 3D-space (6-dof)

**1-dof joints**: Rotary (R) or Prismatic (P) (5 constraints)



N links  
M joints  
→ DOF = 6N - 5M  
→ If N=M, then DOF=N.

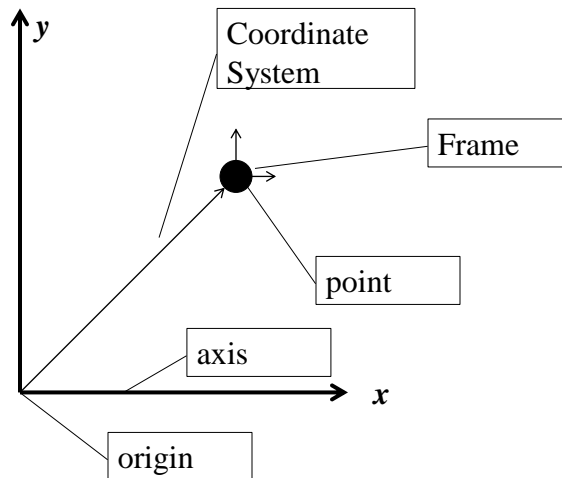


## Kinematics

- Kinematic modelling is one of the most important analytical tools of robotics.
- Used for modelling mechanisms, actuators and sensors
- Used for on-line control and off-line programming and simulation
- In mobile robots kinematic models are used for:
  - steering (control, simulation)
  - perception (image formation)
  - sensor head and communication antenna pointing
  - world modelling (maps, object models)
  - terrain following (control feedforward)
  - gait control of legged vehicles

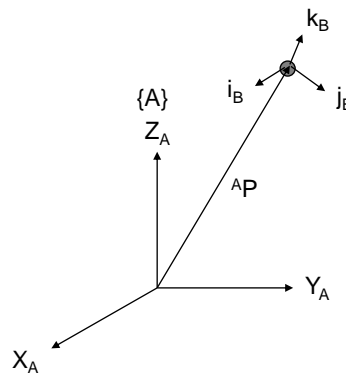


## Basic Terminology



## Coordinate System

- The position and orientation as specified only make sense with respect to some coordinate system



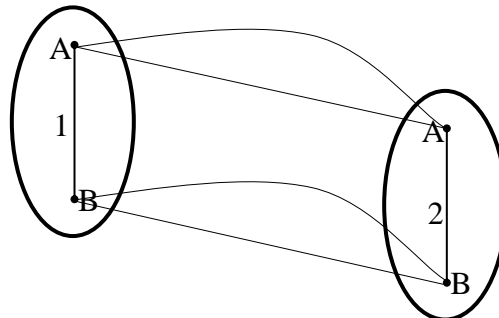
## Frames of Reference

- A frame of reference defines a coordinate system relative to some point in space
- It can be specified by a position and orientation relative to other frames
- The *inertial frame* is taken to be a point that is assumed to be fixed in space
- Two types of motion:
  - Translation
  - Rotation



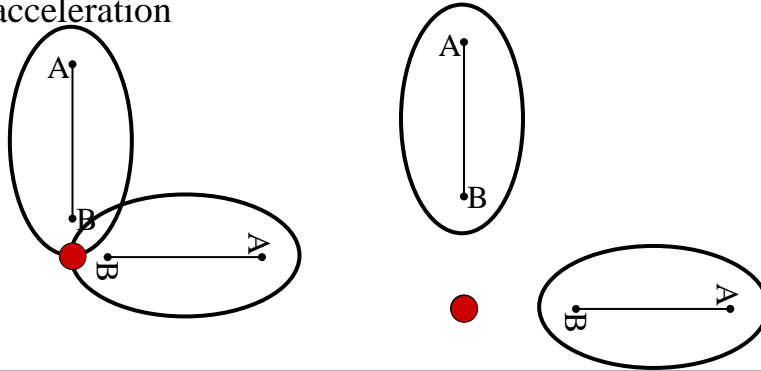
## Translation

- A motion in which a straight line within the body keeps the same direction during the
  - **Rectilinear Translation:** Along straight lines
  - **Curvilinear Translation:** Along curved lines



## Rotation

- The particles forming the rigid body move in parallel planes along circles centered around the same fixed axis (called the **axis of rotation**).
- Points on the axis of rotation have zero velocity and acceleration



## Rotation: Representations

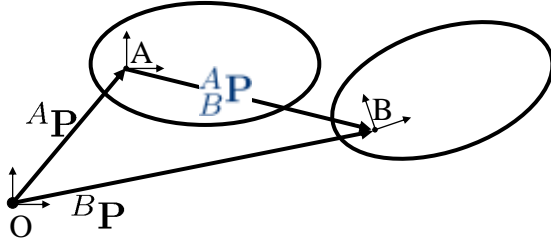
- Orientation are not “Cartesian”
  - Non-commutative
  - Multiple representations
- Some representations:
  - **Rotation Matrices:** Homogenous Coordinates
  - Euler Angles: 3-sets of rotations in sequence
  - Quaternions: a 4-parameter representation that exploits  $\frac{1}{2}$  angle properties
  - Screw-vectors (from Charles Theorem) : a canonical representation, its reciprocal is a “wrench” (forces)



## Position and Orientation [1]

- A **position** vectors specifies the location of a **point** in 3D (Cartesian) space

$$\mathbf{P} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$



$$A_P + A_{PB} - B_P = 0$$

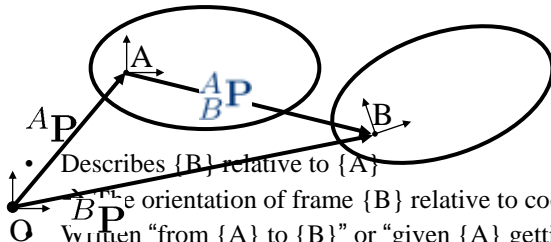
$$A_{PB} = A_{P_B} = A_B P = \begin{bmatrix} B_{p_x} \\ B_{p_y} \\ B_{p_z} \end{bmatrix} - \begin{bmatrix} A_{p_x} \\ A_{p_y} \\ A_{p_z} \end{bmatrix}$$

- BUT we **also** concerned with its orientation in 3D space.  
This is specified as a matrix based on each **frame's unit vectors**



## Position and Orientation [2]

- Orientation in 3D space:  
This is specified as a matrix based on each **frame's unit vectors**



- Describes {B} relative to {A}
- The orientation of frame {B} relative to coordinate frame {A}
- Written "from {A} to {B}" or "given {A} getting to {B}"

$$A R_B = A_B R = \begin{bmatrix} A_{\hat{i}_B} & A_{\hat{j}_B} & A_{\hat{k}_B} \end{bmatrix}$$

- Columns** are **{B} written in {A}**





## Position and Orientation [3]

- The rotations can be analysed based on the unit components ...
- That is: the components of the orientation matrix are the unit vectors projected **onto** the unit directions of the reference frame

$${}^A_B\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\begin{array}{l} {}^A_B\mathbf{R} \\ (a_x)\hat{i}_A \\ (a_y)\hat{j}_A \\ (a_z)\hat{k}_A \end{array} \begin{array}{l} (b_x)\hat{i}_B \quad (b_y)\hat{j}_B \quad (b_z)\hat{k}_B \\ \left[ \begin{array}{ccc} \hat{i}_B \cdot \hat{i}_A & \hat{j}_B \cdot \hat{i}_A & \hat{k}_B \cdot \hat{i}_A \\ \hat{i}_B \cdot \hat{j}_A & \hat{j}_B \cdot \hat{j}_A & \hat{k}_B \cdot \hat{j}_A \\ \hat{i}_B \cdot \hat{k}_A & \hat{j}_B \cdot \hat{k}_A & \hat{k}_B \cdot \hat{k}_A \end{array} \right] \end{array}$$



## Position and Orientation [4]

- Rotation is orthonormal

$$\begin{array}{l} {}^A_B\mathbf{R} \\ (a_x)\hat{i}_A \\ (a_y)\hat{j}_A \\ (a_z)\hat{k}_A \end{array} \begin{array}{l} (b_x)\hat{i}_B \quad (b_y)\hat{j}_B \quad (b_z)\hat{k}_B \\ \left[ \begin{array}{ccc} \hat{i}_B \cdot \hat{i}_A & \hat{j}_B \cdot \hat{i}_A & \hat{k}_B \cdot \hat{i}_A \\ \hat{i}_B \cdot \hat{j}_A & \hat{j}_B \cdot \hat{j}_A & \hat{k}_B \cdot \hat{j}_A \\ \hat{i}_B \cdot \hat{k}_A & \hat{j}_B \cdot \hat{k}_A & \hat{k}_B \cdot \hat{k}_A \end{array} \right] \end{array}$$

- The of a rotation matrix inverse – the transpose

$$\mathbf{R} \cdot \mathbf{R}^T = \mathbf{1}$$

→ thus, the **rows** are **{A} written in {B}**

$${}^B_A\mathbf{R} = {}^A_B\mathbf{R}^T = {}^A_B\mathbf{R}^{-1}$$



## Position and Orientation [5]:

### A note on orientations

- Orientations, as defined earlier, are represented by three orthonormal vectors
- Only three of these values are unique and we often wish to define a particular rotation using three values (it's easier than specifying 9 orthonormal values)
- There isn't a unique method of specifying the angles that define these transformations



## Position and Orientation [7]

- Shortcut Notation:

$$\cos(\theta_a) = c\theta_a = c_a$$

$$\sin(\theta_a) = s\theta_a = s_a$$

$$\cos(\theta_a + \theta_b) = c_{ab}$$

$$\therefore s_{ab} = \boxed{\quad ? \quad}$$



## Position and Orientation [8]

- Rotation Formula about the 3 Principal Axes by  $\theta$

$$\text{X:} \quad \mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\text{Y:} \quad \mathbf{R}_y = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$\text{Z:} \quad \mathbf{R}_z = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



## Euler Angles

- Minimal representation of orientation  $(\alpha, \beta, \gamma)$
- Represent a rotation about an axis of a **moving** coordinate frame
  - ${}^A_B\mathbf{R}$ : Moving frame **B** w/r/t fixed A
- The location of the axis of each successive rotation depends on the previous one! ...
- So, Order Matters (12 combinations, why?)
- Often Z-Y-X:
  - $\alpha$ : rotation about the **z** axis
  - $\beta$ : rotation about the rotated **y** axis
  - $\gamma$ : rotation about the twice rotated **x** axis
- Has singularities! ... (e.g.,  $\beta = \pm 90^\circ$ )



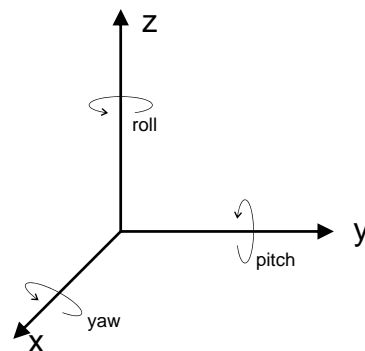
## Fixed Angles

- Represent a rotation about an axis of a **fixed** coordinate frame.
- Again 12 different orders
- Interestingly:  
3 rotations about 3 axes of a **fixed** frame define the same orientation as the same 3 rotations taken in the **opposite order** of the **moving** frame
- For X-Y-Z:
  - $\psi$ : rotation about  $\mathbf{x}_A$  (sometimes called “yaw”)
  - $\theta$ : rotation about  $\mathbf{y}_A$  (sometimes called “pitch”)
  - $\phi$ : rotation about  $\mathbf{z}_A$  (sometimes called “roll”)



## Roll – Pitch – Yaw

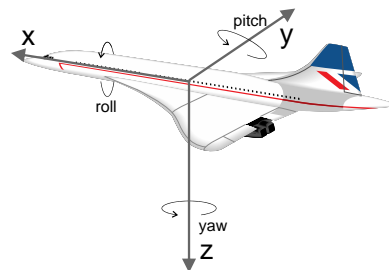
- In many Kinematics References:



→ **Be careful:**

This name is given to other conventions too!

- In many Engineering Applications:



## Euler Angles [1]: X-Y-Z Fixed Angles (Roll-Pitch-Yaw)

- One method of describing the orientation of a Frame {B} is:
  - Start with the frame coincident with a known reference {A}. Rotate {B} first about  $X_A$  by an angle  $\gamma$ , then about  $Y_A$  by an angle  $\beta$  and finally about  $Z_A$  by an angle  $\alpha$ .

$$\begin{aligned}
 {}^A R_{BXYZ}(\gamma, \beta, \alpha) &= R_Z(\alpha)R_Y(\beta)R_X(\gamma) \\
 &= \begin{bmatrix} c_\alpha & -s_\alpha & 0 \\ s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\beta & 0 & s_\beta \\ 0 & 1 & 0 \\ -s_\beta & 0 & c_\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\gamma & -s_\gamma \\ 0 & s_\gamma & c_\gamma \end{bmatrix} \\
 &= \begin{bmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}
 \end{aligned}$$



## Euler Angles [2]: Z-Y-X Euler Angles

- Another method of describing the orientation of {B} is:
  - Start with the frame coincident with a known reference {A}. Rotate {B} first about  $Z_B$  by an angle  $\alpha$ , then about  $Y_B$  by an angle  $\beta$  and finally about  $X_B$  by an angle  $\gamma$ .

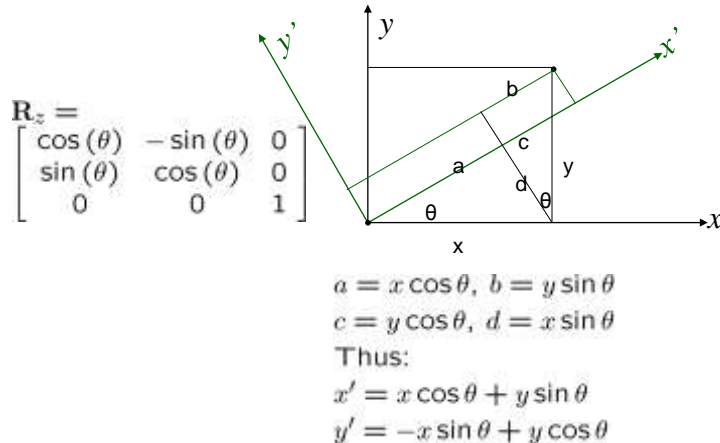
$$\begin{aligned}
 {}^A R_{BZ'Y'X'}(\gamma, \beta, \alpha) &= R_Z(\alpha)R_Y(\beta)R_X(\gamma) \\
 &= \begin{bmatrix} c_\alpha & -s_\alpha & 0 \\ s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\beta & 0 & s_\beta \\ 0 & 1 & 0 \\ -s_\beta & 0 & c_\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\gamma & -s_\gamma \\ 0 & s_\gamma & c_\gamma \end{bmatrix} \\
 &= \begin{bmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}
 \end{aligned}$$



## Position and Orientation [6]:

### “Proof” of Principal Rotation Matrix Terms

- Geometric:



## Unit Quaternion ( $\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3$ ) [1]

- Does not suffer from singularities

$$\epsilon \equiv \epsilon_0 + (\epsilon_1 \hat{i} + \epsilon_2 \hat{j} + \epsilon_3 \hat{k})$$

- Uses a “4-number” to represent orientation

$$ii = jj = kk = -1$$

$$ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j$$

- Product:

$$\begin{aligned}
 \mathbf{ab} = & (a_0b_0 - a_1b_1 - a_2b_2 + a_3b_3) \\
 & + (a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2) \hat{i} \\
 & + (a_0b_2 + a_2b_0 + a_3b_1 + a_1b_3) \hat{j} \\
 & + (a_0b_3 + a_3b_0 + a_1b_2 - a_2b_1) \hat{k}
 \end{aligned}$$

- Conjugate:

$$\tilde{\epsilon} \equiv \epsilon_0 - \epsilon_1 \hat{i} - \epsilon_2 \hat{j} - \epsilon_3 \hat{k}$$

$$\epsilon \tilde{\epsilon} = \tilde{\epsilon} \epsilon = \epsilon_0^2 + \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2$$



## Unit Quaternion [2]: Describing Orientation

- Set  $\epsilon_0 = 0$   
Then  $\mathbf{p} = (p_x, p_y, p_z) \rightarrow \mathbf{p} = p_x \hat{\mathbf{i}} + p_y \hat{\mathbf{j}} + p_z \hat{\mathbf{k}}$
- Then given  $\epsilon$   
the operation  $\epsilon \mathbf{p} \tilde{\epsilon}$  : rotates  $\mathbf{p}$  about  $(\epsilon_1, \epsilon_2, \epsilon_3)$
- Unit Quaternion  $\rightarrow$  Rotation Matrix

$$\mathbf{R} = \begin{pmatrix} 1 - 2(\epsilon_2^2 + \epsilon_3^2) & 2(\epsilon_1\epsilon_2 - \epsilon_0\epsilon_3) & 2(\epsilon_1\epsilon_3 - \epsilon_0\epsilon_2) \\ 2(\epsilon_1\epsilon_2 - \epsilon_0\epsilon_3) & 1 - 2(\epsilon_1^2 + \epsilon_3^2) & 2(\epsilon_2\epsilon_3 - \epsilon_0\epsilon_1) \\ 2(\epsilon_1\epsilon_3 - \epsilon_0\epsilon_2) & 2(\epsilon_2\epsilon_3 - \epsilon_0\epsilon_1) & 1 - 2(\epsilon_1^2 + \epsilon_2^2) \end{pmatrix}$$



## Direction Cosine

- Uses the Direction Cosines (read dot products) of the Coordinate Axes of the moving frame with respect to the fixed frame  ${}^A\mathbf{u} = u_x \hat{\mathbf{i}} + u_y \hat{\mathbf{j}} + u_z \hat{\mathbf{k}}$   
 ${}^A\mathbf{v} = v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} + v_z \hat{\mathbf{k}}$   
 ${}^A\mathbf{w} = w_x \hat{\mathbf{i}} + w_y \hat{\mathbf{j}} + w_z \hat{\mathbf{k}}$
- It forms a rotation matrix!

$${}^A_B \mathbf{R} = \begin{matrix} (a_x) \hat{\mathbf{i}}_A & (b_x) \hat{\mathbf{i}}_B & (b_y) \hat{\mathbf{j}}_B & (b_z) \hat{\mathbf{k}}_B \\ (a_y) \hat{\mathbf{j}}_A & & & \\ (a_z) \hat{\mathbf{k}}_A & & & \end{matrix} \begin{bmatrix} \hat{\mathbf{i}}_B \cdot \hat{\mathbf{i}}_A & \hat{\mathbf{j}}_B \cdot \hat{\mathbf{i}}_A & \hat{\mathbf{k}}_B \cdot \hat{\mathbf{i}}_A \\ \hat{\mathbf{i}}_B \cdot \hat{\mathbf{j}}_A & \hat{\mathbf{j}}_B \cdot \hat{\mathbf{j}}_A & \hat{\mathbf{k}}_B \cdot \hat{\mathbf{j}}_A \\ \hat{\mathbf{i}}_B \cdot \hat{\mathbf{k}}_A & \hat{\mathbf{j}}_B \cdot \hat{\mathbf{k}}_A & \hat{\mathbf{k}}_B \cdot \hat{\mathbf{k}}_A \end{bmatrix}$$



## Screw Displacements

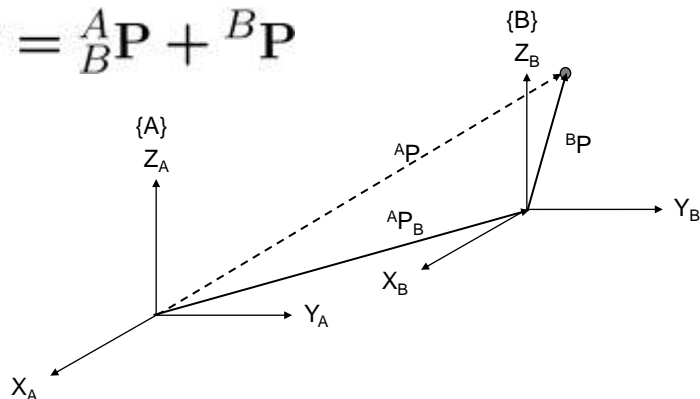
- Comes from the notion that all motion can be viewed as a rotation (Rodrigues formula)
- Define a vector along the axis of motion (screw vector)
  - Rotation (screw angle)
  - Translation (pitch)
  - Summations → via the screw triangle!



## Coordinate Transformations [1]

- Translation Again:  
If {B} is translated with respect to {A} **without rotation**, then it is a vector sum

$${}^A\mathbf{P} = {}^A\mathbf{B}\mathbf{P} + {}^B\mathbf{P}$$





## Coordinate Transformations [2]

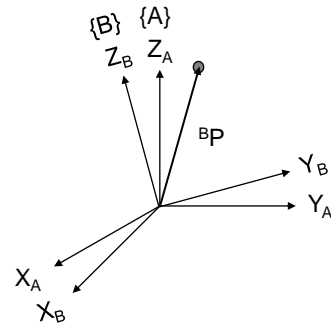
- Rotation Again:  
 {B} is rotated with respect to {A} then  
 use rotation matrix to determine new components

$${}^A P = {}_B^A R {}^B P$$

- NOTE:
  - The Rotation matrix's *subscript* matches the position vector's *superscript*

$${}^A P = {}_B^A R [{}^B] P$$

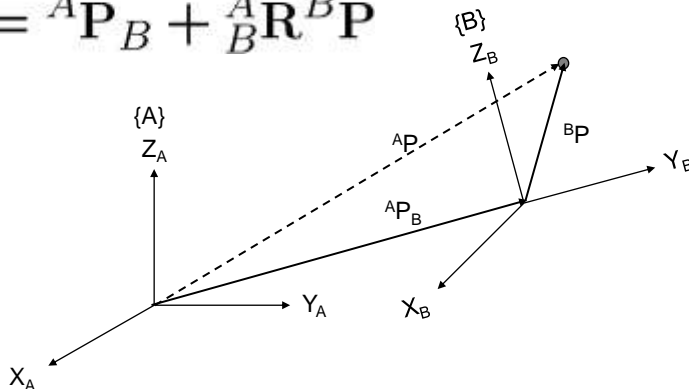
- This gives Point Positions of {B} ORIENTED in {A}



## Coordinate Transformations [3]

- Composite transformation:  
 {B} is moved with respect to {A}:

$${}^A P = {}^A P_B + {}_B^A R {}^B P$$



## General Coordinate Transformations [1]

- A compact representation of the translation and rotation is known as the **Homogeneous Transformation**

$${}^A_B\mathbf{T} = \begin{bmatrix} & {}^A_B\mathbf{R} & & {}^A\mathbf{P}_B \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

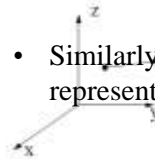
- This allows us to cast the rotation transform in a single matrix form

$$\begin{bmatrix} {}^A\mathbf{P} \\ 1 \end{bmatrix} = {}^A_B\mathbf{T} \begin{bmatrix} {}^B\mathbf{P} \\ 1 \end{bmatrix}$$

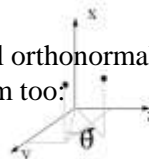


## General Coordinate Transformations [2]

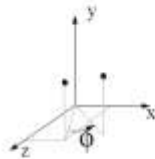
- Similarly, fundamental orthonormal transformations can be represented in this form too:



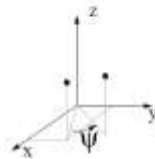
$$\text{Trans}(u, v, w) = \begin{bmatrix} 1 & 0 & 0 & u \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & w \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$\text{Rot}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\theta & -s\theta & 0 \\ 0 & s\theta & c\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$\text{Rot}_y(\phi) = \begin{bmatrix} c\phi & 0 & s\phi & 0 \\ 0 & 1 & 0 & 0 \\ -s\phi & 0 & c\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



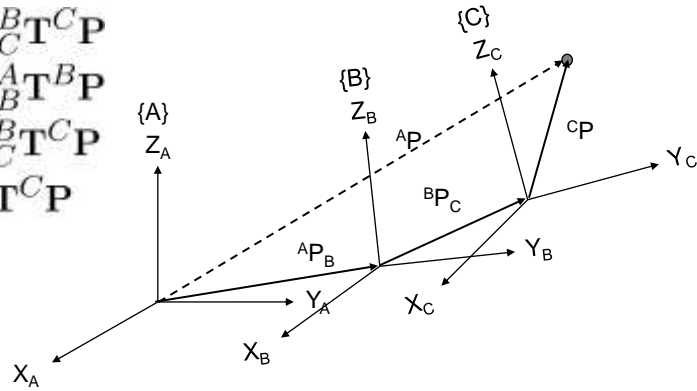
$$\text{Rot}_z(\psi) = \begin{bmatrix} c\psi & -s\psi & 0 & 0 \\ s\psi & c\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



## General Coordinate Transformations [3]

- Multiple transformations compounded as a chain

$$\begin{aligned} {}^B\mathbf{P} &= {}^B\mathbf{T}^C\mathbf{P} \\ {}^A\mathbf{P} &= {}^A\mathbf{T}^B\mathbf{P} \\ &= {}^A\mathbf{T}^B{}^B\mathbf{T}^C\mathbf{P} \\ &= {}^A\mathbf{T}^C\mathbf{P} \end{aligned}$$



$${}^A\mathbf{T}^C = \begin{bmatrix} {}^A\mathbf{R}^B & {}^A\mathbf{P}_B & {}^A\mathbf{R}^B{}^B\mathbf{R}^C & {}^A\mathbf{P}_B + {}^A\mathbf{R}^B{}^B\mathbf{R}^C\mathbf{C}_P \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



## Homogenous Coordinates

$$\hat{\mathbf{p}} = \begin{bmatrix} \rho p_x & \rho p_y & \rho p_z & \rho \end{bmatrix}^T$$

- $\rho$  is a scaling value



## Homogenous Transformation

$$\begin{bmatrix} {}^A R_B & {}^A p \\ \gamma & \rho \end{bmatrix}$$

- $\gamma$  is a projective transformation







## Projective Transformations ...

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, <b>order of contact</b> : intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps, cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $l_\infty$ .
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, <b>I, J</b> (see section 2.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area



## Projective Transformations & Other Transformations of 3D Space

Group	Matrix	Distortion	Invariant properties
Projective 15 dof	$\begin{bmatrix} A & t \\ v^T & p \end{bmatrix}$		Intersection and tangency of surfaces in contact. Sign of Gaussian curvature.
Affine 12 dof	$\begin{bmatrix} A & t \\ 0^T & 1 \end{bmatrix}$		Parallelism of planes; volume ratios, centroids. The plane at infinity, $\pi_{\infty}$ , (see section 3.5).
Similarity 7 dof	$\begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix}$		The absolute conic, $\Omega_{\infty}$ (see section 3.6).
Euclidean 6 dof	$\begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix}$		Volume.

p.78, R. Hartley and A. Zisserman. *Multiple View Geometry in Computer Vision*



## Special Orthogonal & Special Euclidean

- $SO(n)$ : **R**otations

$$SO(n) = \{R \in \mathbb{R}^{n \times n} : RR^T = I, \det R = +1\}$$

$$\exp(\omega) = e^{\omega} = I + R\omega + \frac{R^2}{2!}\omega^2 + \frac{R^3}{3!}\omega^3 + \dots$$

- $SE(n)$ : Transformations of EUCLIDEAN space

$$SE(n) := \mathbb{R}^n \times SO(n)$$

$$SE(3) = \{(p, R) : p \in \mathbb{R}^3, R \in SO(3)\} = \mathbb{R}^3 \times SO(3)$$



## Forward Kinematics [1]

- Forward kinematics is the process of chaining homogeneous transforms together. For example to:
  - Find the articulations of a mechanism, or
  - the fixed transformation between two frames which is known in terms of linear and rotary parameters.
- Calculates the final position from the **machine (joint variables)**
- Unique for an open kinematic chain (**serial arm**)
- “Complicated” (multiple solutions, etc.) for a closed kinematic chain (**parallel arm**)

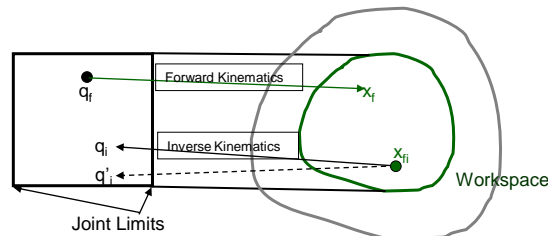


## Forward Kinematics [2]

- Can think of this as “spaces”:
  - Operation space  $(x, y, z, \alpha, \beta, \gamma)$ :  
The robot’s position & orientation
  - Joint space  $(\theta_1 \dots \theta_n)$ :  
A state-space vector of joint variables

$$\vec{x} = \begin{bmatrix} \vec{p} \\ \vec{\Theta} \end{bmatrix}$$

$$\vec{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$



## Summary

- Many ways to view a rotation
  - Rotation matrix
  - Euler angles
  - Quaternions
  - Direction Cosines
  - Screw Vectors
- Homogenous transformations
  - Based on homogeneous coordinates



## Cool Robotics Share

