Introduction to State-Space

or

"States… in… spaaace!"

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Previously on METR4202…

- Robotics, kinematics, perception, oh my!
- You built robot arms to drive an end-effector to a specific point in space – some of you did well!
- …and many of you discovered that that's hard to do accurately!

What you already know*

- Signals can be represented by transfer functions in the s-domain
- Roots of a transfer function’s denominator (poles) indicate the stability of the system
- Poles move around under feedback control
  - Feedback can stabilise an unstable system

*If you have no idea what I’m talking about, now is the time to mention it.

A quick recap

- Differential equations are used to represent the dynamics of systems in time:

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, t)$$

- For linear systems, we use the Laplace transform to represent differential operators:

$$s\boldsymbol{x} = \mathcal{L}\{f(\boldsymbol{x}, t)\}$$

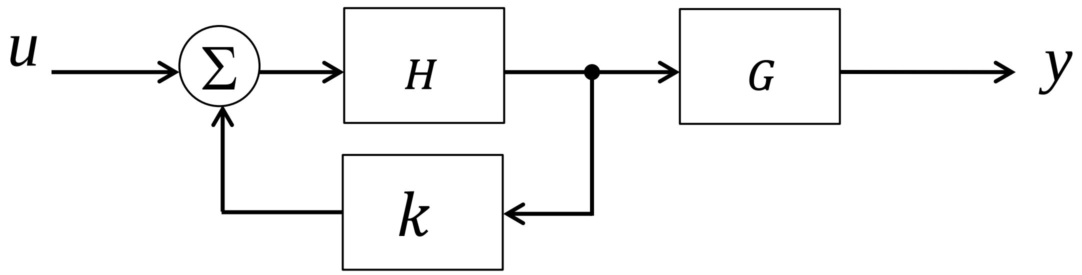
A quick recap

- For SISO LTI* systems, output y is a linear function of input u in the Laplace domain:

$$y = Hu$$

H is the 'transfer function' relating y and u

- We use block diagrams to represent such systems in convenient graphical form:



*Single Input Single Output, Linear Time Invariant

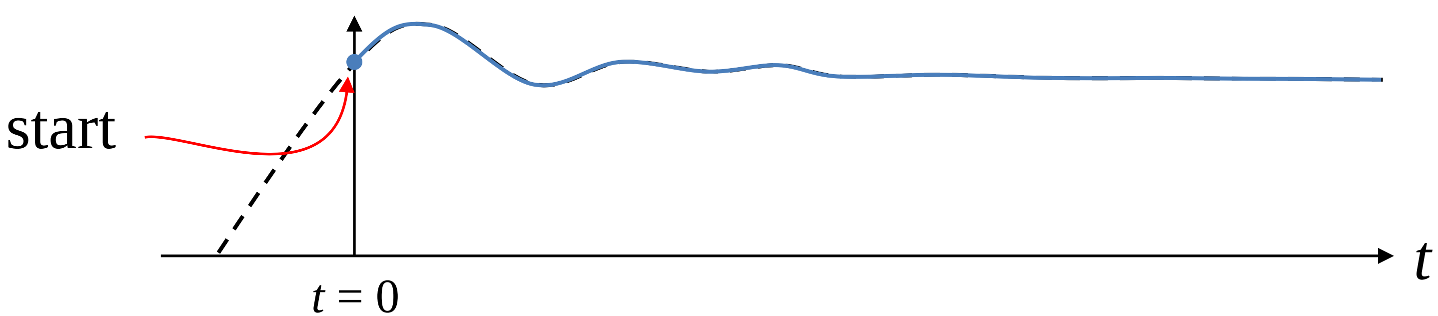
State-space lolwut?

- A ‘clean’ way of representing systems
- Easy implementation in matrix algebra
- Simplifies understanding Multi-Input-Multi-Output (MIMO) systems

Introduction to states*

- Introductory brain-teaser:
  - If you have a step response model of a system with integration, how do you represent non-zero initial conditions?

Eg. how would you setup a simulation of a step response, mid-step?



*Not-insubstantial portions of these slides are based on Franklin, Powel and Enami-Naeni

Store the values

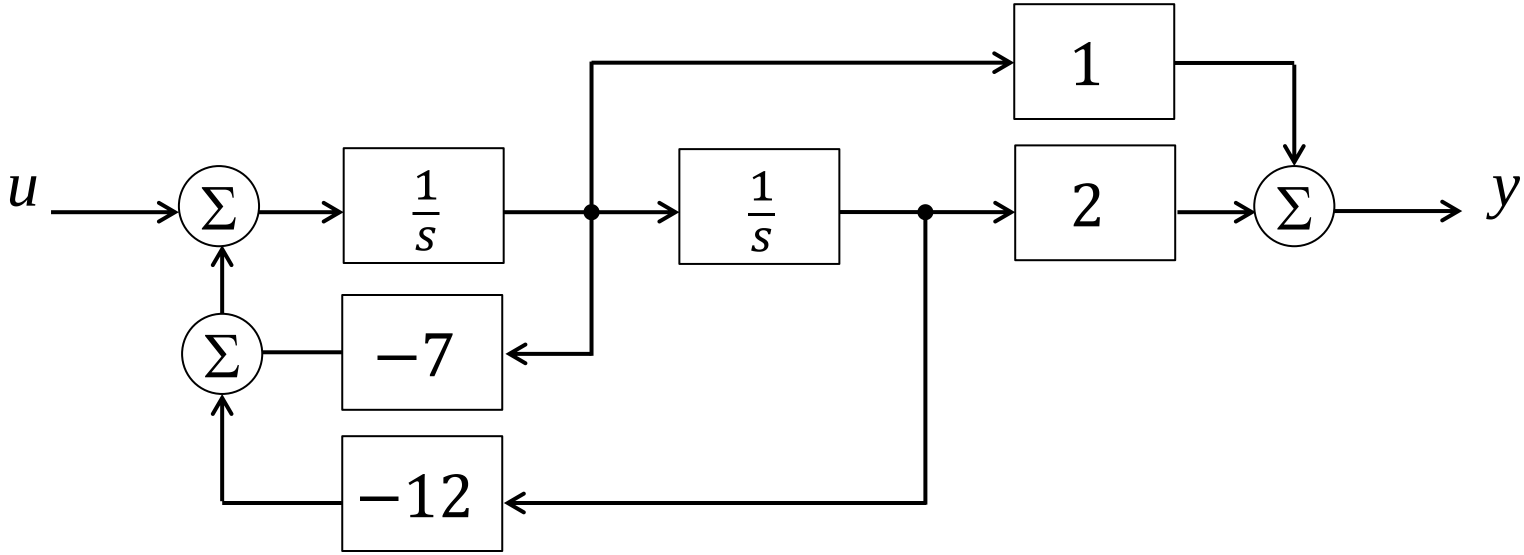
- The time-history of dynamic systems can be encapsulated by ‘states’.
- A state is any previous value upon which future outputs depend:
  – Eg. velocities, altitude, displacement, charge potential, stored magnetic fields, etc.

All the state values of a system are stored in a single column vector x, which is collectively termed “the system’s state”.

Finding states

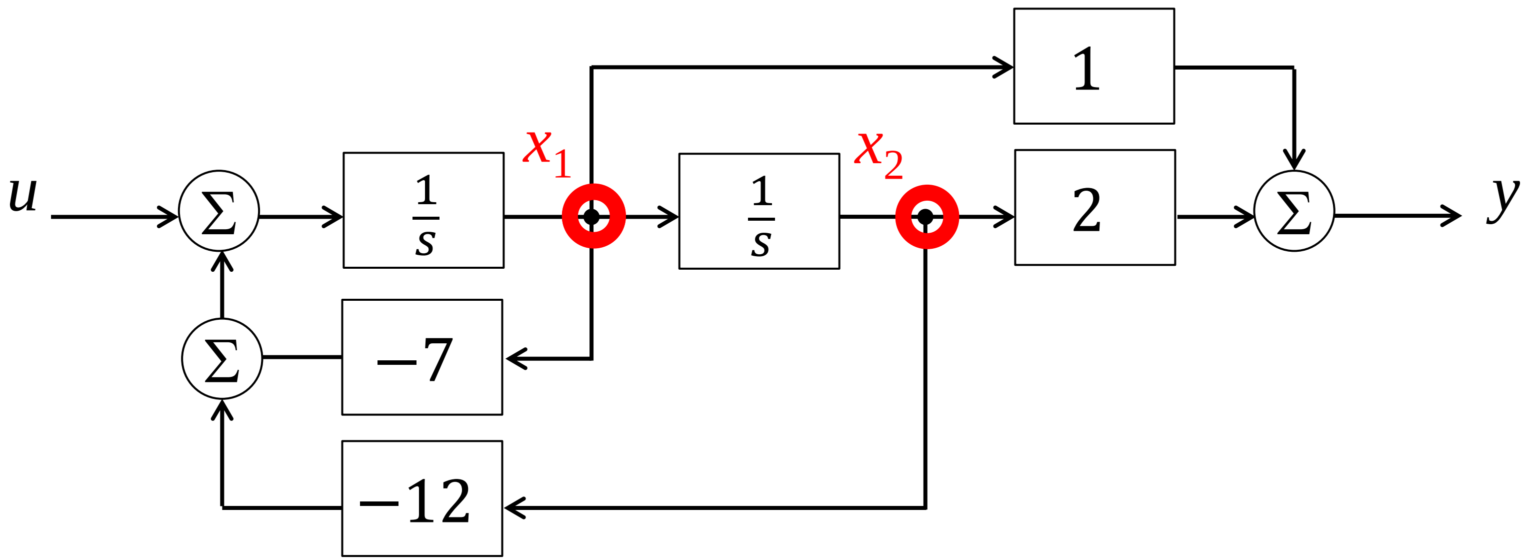
- Linear systems can be written as networks of simple dynamic elements:

$$H = \frac{s+2}{s^2+7s+12} = \frac{2}{s+4} + \frac{-1}{s+3}$$



Finding states

- We can identify the nodes in the system
  - These nodes contain the integrated time-history values of the system response
  - We call them "states"



Linear system equations

- We can represent the dynamic relationship between the states with a linear system:

$$\dot{x}_1 = -7x_1 - 12x_2 + u$$
$$\dot{x}_2 = x_1 + 0x_2 + 0u$$

$$y = x_1 + 2x_2 + 0u$$

State-space representation

- We can write linear systems in matrix form:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -7 & 12 \\ 1 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

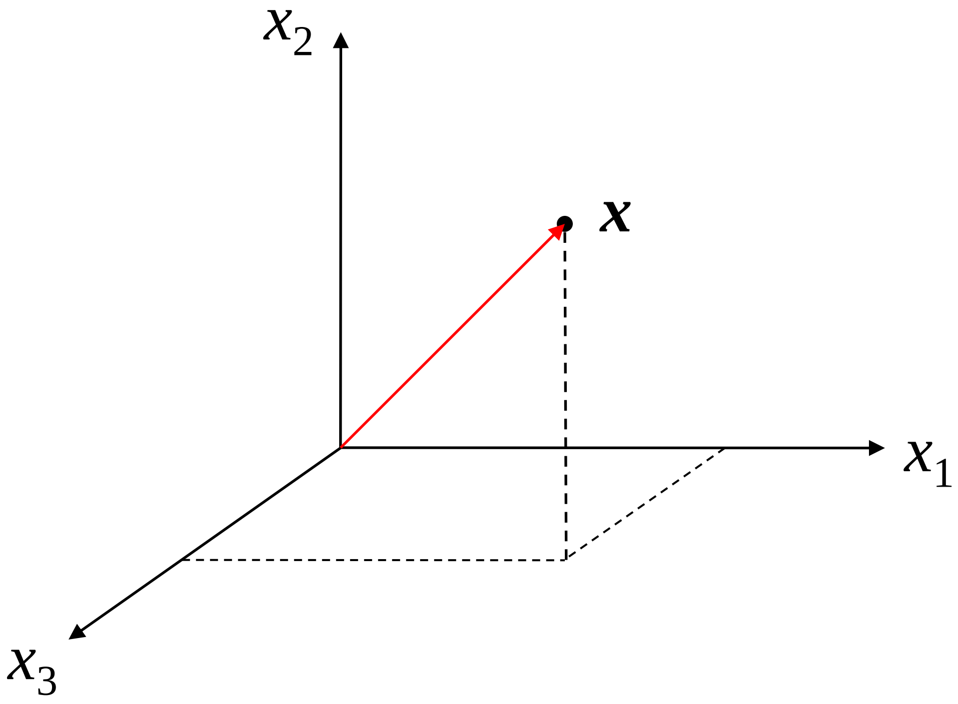
$$\boldsymbol{y} = \begin{bmatrix} 1 & 2 \end{bmatrix} \boldsymbol{x} + 0u$$

Or, more generally:

$$\left.\begin{aligned} \dot{\boldsymbol{x}} &= \mathbf{A}\boldsymbol{x} + \mathbf{B}u \\ y &= \mathbf{C}\boldsymbol{x} + Du \end{aligned}\right\} \text{“State-space equations”}$$

Why "State-space"?

- State vector entries can be thought of as coordinates in a space; hence 'state-space'



eg. $\mathbb{R}^3$

State-space representation

- State-space matrices are not necessarily unique representations of a dynamic system
  - There are several common forms (here's two)
- Control canonical form
  - Each node – each entry in ***x*** – represents a state of the system (each order of *s* maps to a state)
- Modal form
  - Diagonals of the state matrix **A** are the poles ("modes") of the transfer function

Other forms

- There are other representations are useful for other purposes:
  - Observer canonical form
  - Phase variable canonical form
  - Jordan canonical form
  - Etc.

But you don't need to know about those for this course, but check out http://www.ece.rutgers.edu/~gajic/psfiles/canonicalforms.pdf if you're interested!

For now, let's focus on CCF and MF

Control canonical form

- CCF matrix representations have the following structure:

$$\begin{bmatrix} -a_1 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} & -a_n \\ 1 & 0 & & 0 & 0 & 0 \\ 0 & 1 & & & & \\ \vdots & & \ddots & & & \vdots \\ & & & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

Pretty diagonal!

Modal form

- MF matrix representations have the following structure:

$$\begin{bmatrix} -p_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -p_2 & & & & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & & & -p_{n-2} & & 0 \\ 0 & & & & -p_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & 0 & -p_n \end{bmatrix}$$

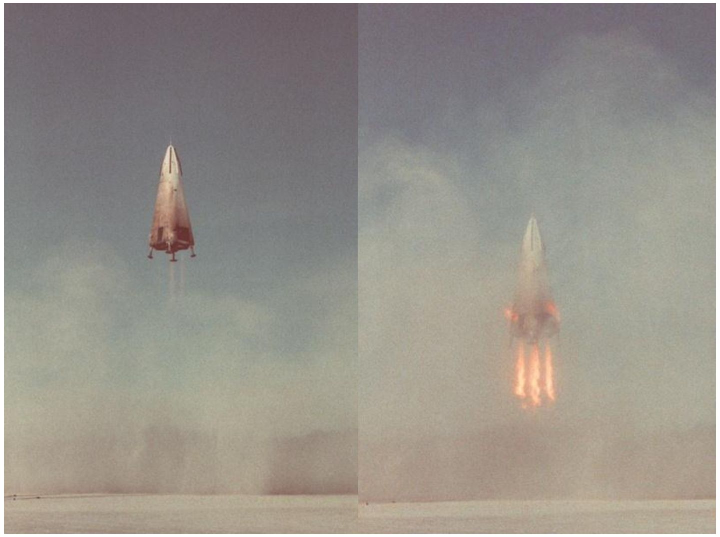
Also pretty diagonal!

EXAMPLE TIME

BREAK TIME

Cool robotics share

The McDonnell Douglas DC-X "Delta Clipper" was a demonstrator developed to explore vertical rocket landing and reusable single-stage to orbit technology. The DC-X used nonlinear state-space control in its flight attitude regulation avionics. Testing of the prototype was carried out by NASA, but the design was a competitor for NASA's own X-33 craft. Despite a rigorous testing schedule, the public success of the DC-X (compared to the embarrassing delays plaguing the X-33) led to its continued development. However, punishing deadlines and burned-out ground crew eventually resulted in a landing accident that severely damaged the craft – the program was quickly cancelled.



DC-X Delta Clipper

State variable transformation

- Important note!
  - The states of a control canonical form system are not the same as the modal states
  - They represent the same dynamics, and give the same output, but the vector values are different!
- However we can convert between them:
  - Consider state representations, $\boldsymbol{x}$ and $\boldsymbol{q}$ where

$$\boldsymbol{x} = \mathbf{T}\boldsymbol{q}$$

$\mathbf{T}$ is a "transformation matrix"

State variable transformation

- Two homologous representations:

$$\begin{aligned}\dot{\boldsymbol{x}} &= \mathbf{A}\boldsymbol{x} + \mathbf{B}u \\ y &= \mathbf{C}\boldsymbol{x} + Du\end{aligned} \quad \text{and} \quad \begin{aligned}\dot{\boldsymbol{q}} &= \mathbf{F}\boldsymbol{q} + \mathbf{G}u \\ y &= \mathbf{H}\boldsymbol{q} + Ju\end{aligned}$$

We can write:

$$\dot{\boldsymbol{x}} = \mathbf{T}\dot{\boldsymbol{q}} = \mathbf{A}\mathbf{T}\boldsymbol{q} + \mathbf{B}u$$
$$\dot{\boldsymbol{q}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\boldsymbol{q} + \mathbf{T}^{-1}\mathbf{B}u$$

Therefore, $\mathbf{F} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ and $\mathbf{G} = \mathbf{T}\mathbf{B}$

Similarly, $\mathbf{C} = \mathbf{H}\mathbf{T}$ and $D = J$

Consider…

- What if we try to turn an arbitrary state description into control canonical form?
  - We expect that for$\mathbf{AT^{-1}} = \mathbf{T^{-1}F}$:

$$\begin{bmatrix} -a_1 & -a_2 & \cdots & -an \\ 1 & 0 & & 0 \\ 0 & \ddots & & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix} = \begin{bmatrix} t_1\mathbf{F} \\ t_2\mathbf{F} \\ \vdots \\ t_n\mathbf{F} \end{bmatrix}$$

where t_i are the rows of $\mathbf{T^{-1}}$

Then, $t_2 = t_3\mathbf{F}$, and $t_1 = t_2\mathbf{F} = t_3\mathbf{F}^2$, etc…

Note: t_n is a row and $t_n\mathbf{F}$ yields a row

Consider…

- $\mathbf{T^{-1}G = B}$, in control canonical form yields

$$\begin{bmatrix} t_1 G \\ t_2 G \\ \vdots \\ t_n G \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The two results together give:

$$t_n\mathbf{G} = 0$$

$$\dots$$

$$t_2\mathbf{G} = t_n\mathbf{F}^{n-1}\mathbf{G} = 0$$

$$t_1\mathbf{G} = t_n\mathbf{F}^{n}\mathbf{G} = 1$$

Consider…

- Look at the last entry for t_3…
  - We can write this as:

$$t_3[\mathbf{G} \quad \mathbf{FG} \quad \dots \quad \mathbf{F}^n\mathbf{G}] = [0 \quad 0 \quad \dots \quad 1]$$

Or

$$t_3 = [0 \quad 0 \quad \dots \quad 1]\boldsymbol{\mathcal{C}}^{-1}$$

where $\boldsymbol{\mathcal{C}} = [\mathbf{G} \quad \mathbf{FG} \quad \mathbf{F}^2\mathbf{G} \quad \cdots \quad \mathbf{F}^{n-1}\mathbf{G}]$

This is called the “controllability matrix”

Controllability matrix

- To convert an arbitrary state representation in **F**, **G**, **H** and J to control canonical form **A**, **B**, **C** and D, the controllability matrix

$$\mathcal{C} = [\mathbf{G} \quad \mathbf{FG} \quad \mathbf{F}^2\mathbf{G} \quad \cdots \quad \mathbf{F}^{n-1}\mathbf{G}]$$

must be invertible (i.e. full rank).

>deep think<

Why is it called the “controllability” matrix?

Controllability matrix

- If you can write it in CCF, then the system equations must be linearly independent.
- Transformation by any invertible matrix preserves the controllability of the system.
- Thus, a invertible controllability matrix means $\boldsymbol{x}$ can be driven to any value.

Kind of awesome

- The controllability of a system depends on the particular set of states you chose
- You can't tell just from a transfer function whether all the states of $\boldsymbol{x}$ are controllable
- System poles are the Eigenvalues of $\mathbf{F}$, (p_i)

State evolution

- Consider the system matrix relation:

$$\dot{\boldsymbol{x}} = \mathbf{F}\boldsymbol{x} + \mathbf{G}u$$
$$y = \mathbf{H}\boldsymbol{x} + Ju$$

The time solution of this system is:

$$\boldsymbol{x}(t) = e^{\mathbf{F}(t-t_0)}\,\boldsymbol{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{F}(t-\tau)}\mathbf{G}u(\tau)d\tau$$

If you didn't know, the matrix exponential is:

$$e^{\mathbf{K}t} = \mathbf{I} + \mathbf{K}t + \frac{1}{2!}\mathbf{K}^2 t^2 + \frac{1}{3!}\mathbf{K}^3 t^3 + \cdots$$

Stability

- We can solve for the natural response to initial conditions $\boldsymbol{x}_{\mathbf{0}}$:

$$\boldsymbol{x}(t) = e^{p_i t}\boldsymbol{x}_0$$

$$\therefore \dot{\boldsymbol{x}}(t) = p_i e^{p_i t}\boldsymbol{x}_0 = \boxed{\mathbf{F}e^{p_i t}\boldsymbol{x}_0}$$

homogenous response

Clearly, a system will be stable provided

$$\mathrm{eig}(\mathbf{F}) < 0$$

Characteristic polynomial

- From this, we can see $\mathbf{F}\boldsymbol{x}_0 = p_i\boldsymbol{x}_0$

$$\text{or, } (p_i\mathrm{I} - \mathbf{F})\boldsymbol{x}_0 = 0$$

$$\text{which is true only when } \det(p_i\mathrm{I} - \mathbf{F})\boldsymbol{x}_0 = 0$$

Aka. the characteristic equation!

- We can reconstruct the CP in s by writing:

$$\det(s\mathrm{I} - \mathbf{F})\boldsymbol{x}_0 = 0$$

Great, so how about control?

- Now that we have a state space model, how do we make the system stable, or converge to desired states?

Easy: Feedback!

- Given $\dot{\boldsymbol{x}} = \mathbf{F}\boldsymbol{x} + \mathbf{G}u$, if we know $\mathbf{F}$ and $\mathbf{G}$, we can design a controller $u = -\mathbf{K}\boldsymbol{x}$ such that

$$\mathrm{eig}(\mathbf{F} - \mathbf{G}\mathbf{K}) < 0$$

Full state feedback

- If we have full measurement and control of the states of $\boldsymbol{x}$, we can position poles of the closed-loop system in arbitrary locations
  - Modal form makes this straight forward:

$$\mathbf{F} - \mathbf{GK} = \begin{bmatrix} -p_1 - G \cdot K_{1j} & & \\ & -p_2 - G \cdot K_{2j} & \\ & & -p_3 - G \cdot K_{3j} \end{bmatrix}$$

Of course, this hardly ever happens in reality.

Brain teaser: Why?

Example: PID control

- Consider a system parameterised by three states: x_1, x_2, x_3 where $x_2 = \dot{x}_1$ and $x_3 = \dot{x}_2$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -2 \end{bmatrix} \boldsymbol{x} - \mathbf{K}u$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \boldsymbol{x} + 0u$$

x_2 is the output state of the system; x_1 is the value of the integral; x_3 is the velocity.

Example: PID control

- We can choose **K** to move the eigenvalues of the system as desired:

$$\det\begin{bmatrix} 1-K_1 & & \\ & 1-K_2 & \\ & & -2-K_3 \end{bmatrix} = \mathbf{0}$$

All of these eigenvalues must be positive.

It's straightforward to see how adding derivative gain K_3 can stabilise the system.

In reality…

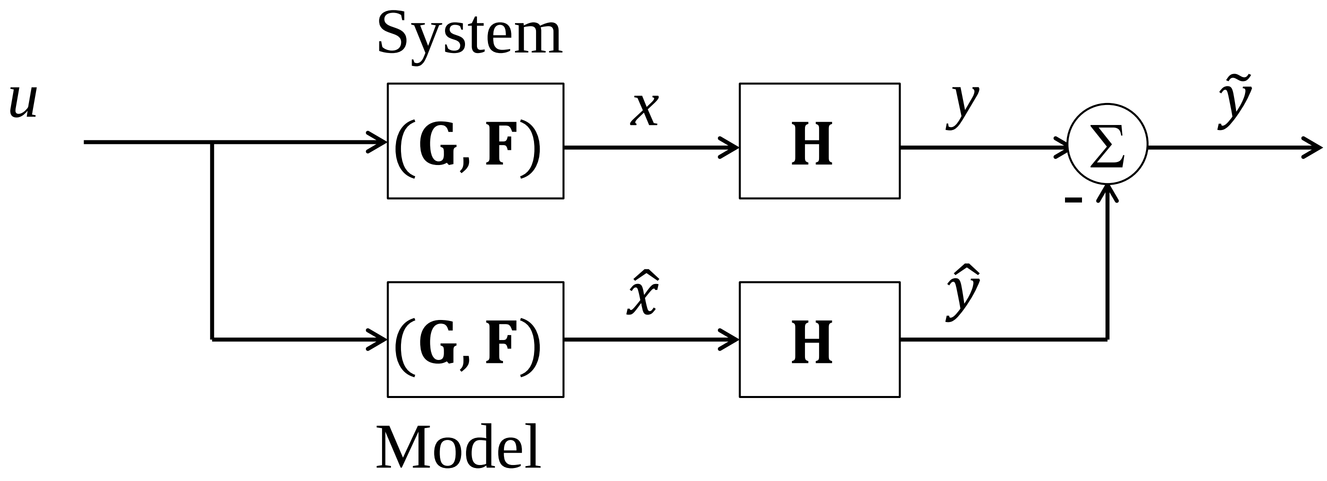
- You can never measure or apply control action to all states directly.
  – The majority of system states will be hidden to the control engineer.

But we can pretend!

- We can design a controller as if we did, using an estimate – an educated guess.

Observers

- Observers (aka "estimators") are used to infer the hidden states of a system from measured outputs.



A controller is designed using estimates in lieu of full measurements

Observers

- The state estimate can be treated like a control system itself
  - Dynamics to update the estimate:
    $$\dot{\hat{x}} = \mathbf{F}\hat{x} + \mathbf{G}\mathrm{u}$$
  - By measuring an 'error signal' from the difference between the real output measurement and the output estimate, $\tilde{x} = x - \hat{x}$, the state estimate can be shown to converge

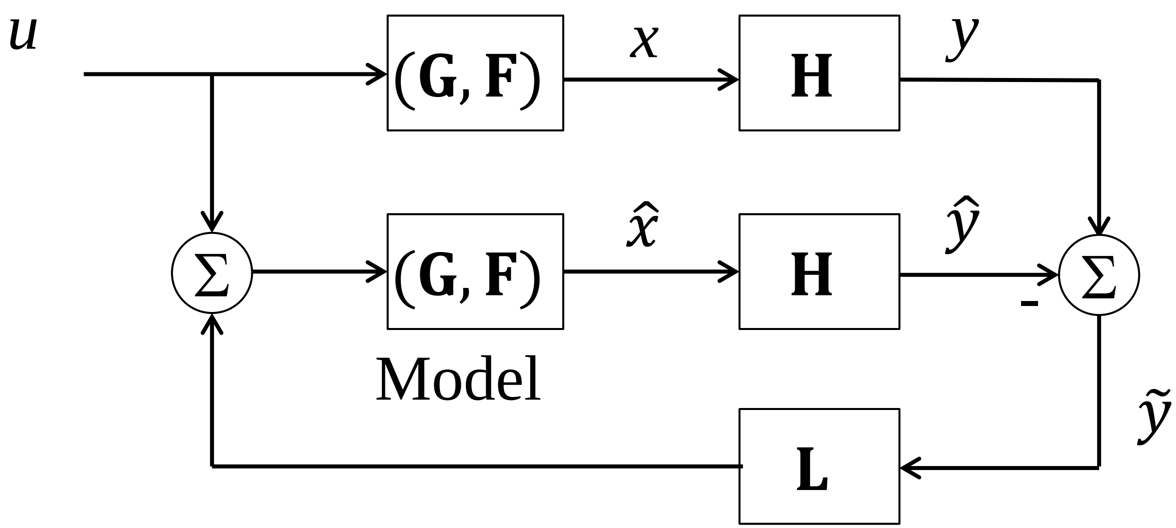
Observers

- Just like you might expect:

$$\dot{\hat{x}} = \mathbf{F}\hat{x} + \mathbf{G}u + \mathbf{L}(y - \mathbf{H}\hat{x})$$

$$\therefore \dot{\tilde{x}} = (\mathbf{F} - \mathbf{LH})\tilde{x}$$

Choose **L** to make $\tilde{x}$ converge to 0



Observability

- The ability to infer these values is called "Observability"
  - This is the dual of controllability; a system that is observable is also controllable and vice versa.
  - Observability matrix:

$$\boldsymbol{\mathcal{O}} = \begin{bmatrix} \mathbf{H} \\ \mathbf{HF} \\ \vdots \\ \mathbf{HF}^{\mathrm{n}-1} \end{bmatrix}$$

Just scratching the surface

- There is a lot of stuff to state-space control
- One lecture (or even two) can't possibly cover it all in depth

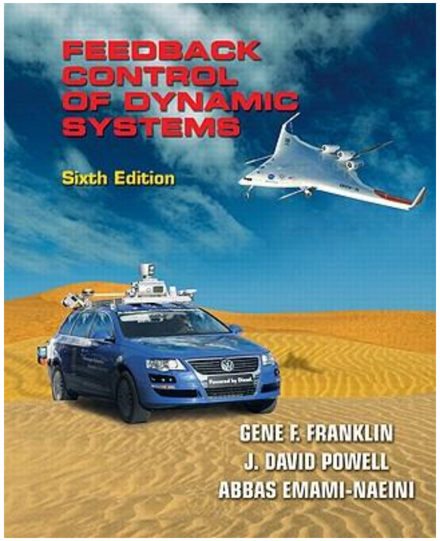
Go play with Matlab and check it out!

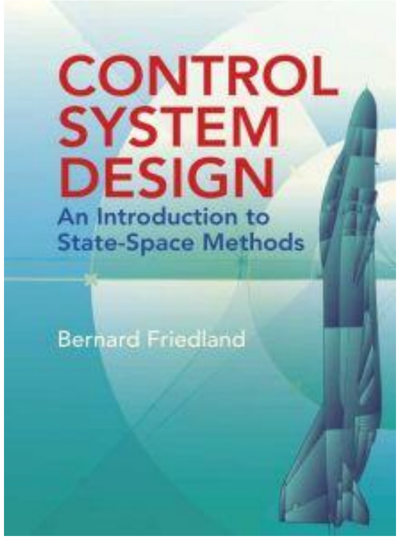
EXAMPLE TIME

Quick plug*

"**Feedback Control of Dynamic Systems**" by Franklin, Powell and Emami-Naeini.

"**Control System Design: An Introduction to State-Space Methods**" by Friedland





* No, they're not paying me – they're just really good books!

Tune-in next time for…

Control/Planning Under Uncertainty

Starring

Hanna Kurniawati!

Fun fact: Saying "eigenvalue" makes you feel smarter!

. . .

And now a word from our hideous sponsor!

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Discretisation FTW!

- We can use the time-domain representation to produce difference equations!

$$\boldsymbol{x}(kT+T) = e^{\mathbf{F}T}\,\boldsymbol{x}(kT) + \int_{kT}^{kT+T} e^{\mathbf{F}(kT+T-\tau)}\mathbf{G}u(\tau)d\tau$$

Notice $\boldsymbol{u}(\tau)$ is not based on a discrete ZOH input, but rather an integrated time-series.

We can structure this by using the form:

$$u(\tau) = u(kT), \qquad kT \le \tau \le kT+T$$

Discretisation FTW!

- Put this in the form of a new variable:

$$\eta = kT + T - \tau$$

Then:

$$\boldsymbol{x}(kT + T) = e^{\boldsymbol{F}T}\boldsymbol{x}(kT) + \left(\int_{kT}^{kT+T} e^{\boldsymbol{F}\eta}\,d\eta\right)\boldsymbol{G}u(kT)$$

Let's rename $\boldsymbol{\Phi} = e^{\boldsymbol{F}T}$ and $\boldsymbol{\Gamma} = \left(\int_{kT}^{kT+T} e^{\boldsymbol{F}\eta}\,d\eta\right)\boldsymbol{G}$

Discrete state matrices

So,

$$\boldsymbol{x}(k+1) = \mathbf{\Phi}\boldsymbol{x}(k) + \mathbf{\Gamma}u(k)$$
$$y(k) = \mathbf{H}\boldsymbol{x}(k) + \mathbf{J}\boldsymbol{u}(k)$$

Again, $\boldsymbol{x}(k+1)$ is shorthand for $\boldsymbol{x}(kT+T)$

Note that we can also write $\mathbf{\Phi}$ as:

$$\mathbf{\Phi} = \mathbf{I} + \mathbf{F}T\mathbf{\Psi}$$

where

$$\mathbf{\Psi} = \mathbf{I} + \frac{\mathbf{F}T}{2!} + \frac{\mathbf{F}^2T^2}{3!} + \cdots$$

Simplifying calculation

- We can also use $\mathbf{\Psi}$ to calculate $\mathbf{\Gamma}$
  - Note that:

$$\Gamma = \sum_{k=0}^{\infty} \frac{\mathbf{F}^k T^k}{(k+1)!} T\mathbf{G}$$
$$= \mathbf{\Psi} T \mathbf{G}$$

$\mathbf{\Psi}$ itself can be evaluated with the series:

$$\mathbf{\Psi} \cong \mathbf{I} + \frac{\mathbf{F}T}{2}\left\{\mathbf{I} + \frac{\mathbf{F}T}{3}\left[\mathbf{I} + \cdots \frac{\mathbf{F}T}{n-1}\left(\mathbf{I} + \frac{\mathbf{F}T}{n}\right)\right]\right\}$$

State-space z-transform

We can apply the z-transform to our system:

$$(z\mathbf{I} - \mathbf{\Phi})\boldsymbol{X}(z) = \mathbf{\Gamma}U(k)$$

$$Y(z) = \mathbf{H}\boldsymbol{X}(z)$$

which yields the transfer function:

$$\frac{Y(z)}{\boldsymbol{X}(z)} = G(z) = \mathbf{H}(z\mathbf{I} - \mathbf{\Phi})^{-1}\mathbf{\Gamma}$$

State-space control design

- Design for discrete state-space systems is just like the continuous case.
  - Apply linear state-variable feedback:

$$u = -\mathbf{K}\boldsymbol{x}$$

such that $\det(z\mathbf{I} - \boldsymbol{\Phi} + \boldsymbol{\Gamma}\mathbf{K}) = \alpha_c(z)$

where $\alpha_c(z)$ is the desired control characteristic equation

Predictably, this requires the system controllability matrix

$\boldsymbol{\mathcal{C}} = [\boldsymbol{\Gamma} \quad \boldsymbol{\Phi}\boldsymbol{\Gamma} \quad \boldsymbol{\Phi}^2\boldsymbol{\Gamma} \quad \cdots \quad \boldsymbol{\Phi}^{n-1}\boldsymbol{\Gamma}]$ to be full-rank.